Upper bounds for domination number

Jan Mełech

TCS UJ

14.05.2020

Outline

1. What is (a, b)-domination set for graph
2. Upper bounds for size of minimal (a, b)-domination set
   - using Turan's Theorem
   - using Lovasz Local Lemma

Definition

Let G be a connected graph of order n with vertex set $V(G)$.
A subset $S \subseteq V(G)$ is an (a, b)-dominating set if every vertex $v \in S$ is adjacent to at least a vertices in S and every $v \in V \setminus S$ is adjacent to at least b vertices in S.
The minimum cardinality of an (a, b)-dominating set of G is the (a, b)-domination number of G, denoted by $\gamma_{a,b}(G)$.

Strategy for obtaining upper bounds using $\delta(G)$

One approach to achieve non-trivial upper bound for domination number is to relate it with $\delta(G)$ - minimum degree of vertices of G.
Strategy for computing an upper bound for (a, b)-domination number of a given graph G with $\delta(G) \geq \max\{a, b\}$ is as follows:

- For a given graph G, we construct another graph G' with the same set of vertices and the following property - for any independent set A of vertices for G', $V(G) \setminus A$ is an (a, b)-dominating set for G.

Therefore, to obtain upper bound for size of (a, b)-dominating set, we need to compute lower bound for the independence number of G.

Turan's Theorem

Theorem (Turan's Theorem)

Let G be any graph with n vertices, such that G is K_{r+1} -free. Then, the number of edges in G is at most $(1 - \frac{1}{r})\frac{n^2}{2}$.

Lower bound for independence number

Turan's Theorem will be helpful in proving the following lemma:

Lemma

Let G be a graph with n vertices and at most αn edges, where $\alpha > 0$ is a fixed number. Then, the independence number of G is at least $\frac{n}{2\alpha+1}$.

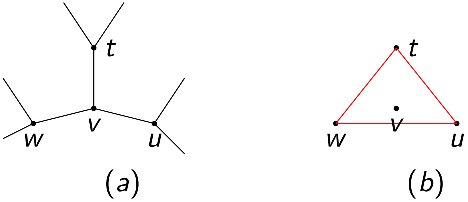
Let r be the biggest integer, so that $r < \frac{n}{2\beta+1}$.
Then $(1 - \frac{1}{r})\frac{n^2}{2} < (1 - \frac{2\beta+1}{n})\frac{n^2}{2} = \binom{n}{2} - \beta n$.
Since the number of edges of the complement of G is at least $\binom{n}{2} - \beta n$, by Turán's theorem, the complement of G has a clique of size $r + 1$.
Hence G has an independent set of size at least $r + 1$.

Theorem

If G is a graph with $\delta(G) \geq 3$ and n vertices, then $\gamma_{2,2}(G) \leq \frac{6}{7}n$. If $\delta(G) \geq 4$, then $\gamma_{2,2}(G) \leq \frac{4}{5}n$.

We construct a graph G' from G as follows:

- The vertices of G' are the same as G
- When $\delta(G) \geq 3$, for each vertex v, choose three of its neighbors and join each pair of them with three edges. This way we get a graph of at most $3n$ edges.



- When $\delta(G) \geq 4$, for each vertex v choose 4 of its neighbors arbitrarily and join them with 2 disjoint edges. This way we get a graph G' of at most $2n$ edges.

In both cases the following statement is true - for any independence set A of G', $V(G) \setminus A$ is a $(2,2)$-dominating set for G.

Due to construction of graph G':

- When $\delta(G) \geq 3$, G' has at most $3n$ edges.
- When $\delta(G) \geq 4$, G' has at most $2n$ edges.

Thus, according to Lemma:

- When $\delta(G) \geq 3$, the independence number of G' is at least $\frac{n}{7}$. So $\gamma_{2,2}(G) \leq \frac{6}{7}n$.
- When $\delta(G) \geq 4$, the independence number of G' is at least $\frac{n}{5}$ and hence $\gamma_{2,2}(G) \leq \frac{4}{5}n$.

Generalization

Above approach can be easily generalized:

Theorem

If G is a graph with n vertices then the following claims are true:

- *If $\delta(G) \geq k+1$, then $\gamma_{k,k}(G) \leq \frac{k(k+1)}{k(k+1)+1}$*
- *If $\delta(G) \geq 2k$ then $\gamma_{k,k}(G) \leq \frac{2k}{2k+1}n$.*

Well-known bounds related to $\delta(G)$

We can compare above results with well-known bound from other publications:
Let $\tilde{d}_m = \frac{1}{n}\sum_{i=1}^{n}\binom{d(v_i)+1}{m}$ and $\hat{d}_m = \frac{1}{n}\sum_{i=1}^{n}\binom{d_i}{m}$. Then we have:

Theorem

[2] For any graph G of minimum degree δ with $1 \leq k \leq \delta + 1$

$$\gamma_{k-1,k}(G) \leq \frac{\ln(\delta - k + 2) + \ln \tilde{d}_{k-1} + 1}{\delta - k + 2} n.$$

Theorem

[3] If k is a positive integer and G is a graph of order n with $\delta > k \geq 1$, then

$$\gamma_{k,k}(G) \leq \frac{\ln(\delta - k) + \ln \hat{d}_k + 1}{\delta - k} n.$$

Lovasz Local Lemma

Second tool useful in deriving upper bound for domination number is Lovasz Local Lemma:

Lemma (Lovasz Local Lemma)

Let $A_1, A_2, ..., A_n$ be a sequence of events such that each event occurs with probability at most p and such that each event is independent of all the other events except for at most d of them. If $epd \leq 1$, then there is a nonzero probability that none of the events occurs.

One of the main application of Lovasz Local Lemma is in proving existence of some given combinatorial structures, what will be also true in case of (a, b)-domination sets.

Upper bound using Lovasz Local Lemma

We want to prove the following claim:

Theorem

For any a, b and $\frac{1}{2} \leq \alpha < 1$, there is $r_0 > 0$ such that, if $r \geq r_0$ and G is an r-regular graph, $\gamma_{a,b}(G) \leq \alpha n$.

Let $N \geq 2$ be a number that $1 - \frac{1}{N} \leq \alpha$.
Color the vertices of G with N colors, such that each color is used randomly and uniformly with probability $\frac{1}{N}$.
We claim that for r large enough, removing any fixed color from the set of vertices will give an (a, b)-dominating set with a positive probability.
So one can get an (a, b)-dominating set of size at most $(1 - \frac{1}{N})n$ by removing a color with highest number of appearances.

Let A_i - event that vertex v_i fails to satisfy the (a, b)-domination property for the set of vertices with one color removed.
Then

$$P(A_i) = \frac{\sum_{i=1}^{a} \binom{r}{a-i}(N-1)^{a-i+1}}{N^r} + \frac{\sum_{i=1}^{b} \binom{r}{b-i}(N-1)^{b-i}}{N^r}$$

First factor in the formula comes from the case when v_i is in the dominating set, then it must have fewer than a neighbors in the dominating set. That is at least $r - a + 1$ of its neighbors must have the same color but different from the color of v_i. Explanation for second factor is analogous.

On the other hand, each event A_i depends on it's neighbors and neighbors of it's neighbors. Which means A_i depends on at most r^2 of the events. Hence by Lovász local lemma, we need to show that to show that for r large enough $eP(A_i)r^2 \leq 1$ for each vertex i.
But since $P(A_i)$ is a a polynomial function in r divided by an exponential function in r, clearly for r large enough the probability approaches zero and hence the condition of Lovász local lemma will be satisfied.

Note that in our computation we have assumed that the graph G is regular. It can be easily seen that for arbitrary graphs with minimum degree δ and maximum degree Δ, in calculating $P(A_i)$ instead of r we can use δ and each $P(A_i)$ depends on at most Δ^2 of the vertices.
So, for the cases when the graph is not regular we can use this formula again.

Table 1: Upper bounds for $\gamma_{a,b}(G)$ achieved by Lovász local lemma for various values of a, b, δ and Δ

δ	Δ	a	b	Upper bound
7	7	2	2	$3/4n$
7	8	2	2	$3/4n$
9	9	2	2	$2/3n$
9	10	2	2	$2/3n$
9	11	2	2	$2/3n$
14	14	2	2	$1/2n$
8	8	1	2	$2/3n$
8	9	1	2	$2/3n$
8	10	1	2	$2/3n$
8	11	1	2	$2/3n$

Bibliography

- Alipour, Sharareh, and Amir Jafari. "Upper Bounds for the Domination Numbers of Graphs Using Turán's Theorem and Lovász Local Lemma".
- Chang, Gerard J. "The upper bound on k-tuple domination numbers of graphs." European Journal of Combinatorics 29, no. 5: 1333-1336 (2008).
- Kazemi, Adel P. "A note on the k-tuple total domination number of a graph." Tbilisi Mathematical Journal 8, no. 2 (2015).