THE 1-2-3 CONJECTURE

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Plan

1. What 1-2-3 conjecture is?
2. Versions of 1-2-3 Conjecture we'll discuss
3. Proof of one of the variants
4. List versions of 1-2-3 Conjecture we'll discuss
5. Another variant proof
6. It's time for some theory
7. Related problems
8. Finish

1-2-3 Conjecture - Base

Edge decorations

Let G be a simple graph. Suppose that each edge e in G is assigned a real number f(e). For each vertex v, let S(v) denote the sum of numbers assigned to the edges incident to v, that is,

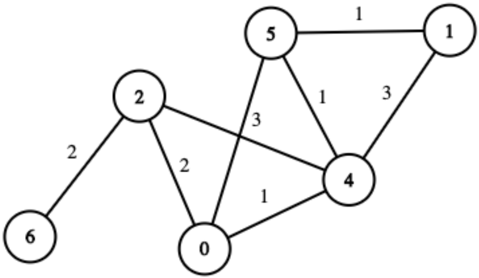
$$S(v) = \sum_{x \in N(v)} f(xv)$$

where N(v) is the set of neighbors of v.
We say that f is a **cool decoration** of the **edges** of G if $S(u) \neq S(v)$ for every pair of adjacent vertices in G.

Conjecture 1 (The 1-2-3 Conjecture)

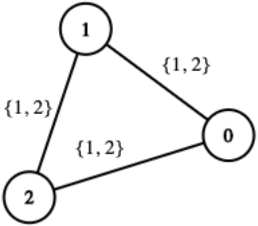
Every connected graph with at least two edges has a cool edge decoration from the set $\{1, 2, 3\}$

1-2-3 Conjecture - Base



1-2-3 Conjecture - Base

Counterexample for set $\{1, 2\}$



1-2-3 Conjecture - versions

Problems (non-list)				
name	type	S function	statement	status
edge (1-2-3)	C1	$S(v) = \sum_{x \in N(v)} f(xv)$	G with $\lvert E \rvert > 1$ has decoration from $\{1, 2, 3\}$.	open
vertex	C2	$S(v) = \sum_{x \in N(v)} f(x)$	G has decoration from $\{1, 2, ..., \chi(G)\}$.	open
total (1-2)	C3	$S(v) = f(v) + \sum_{x \in N(v)} f(vx)$	G has decoration from $\{1, 2\}$.	open
weaker total (1-2)	T1		G has decoration from $\{1, 2\}$ for V and $\{1, 2, 3\}$ for E.	proof

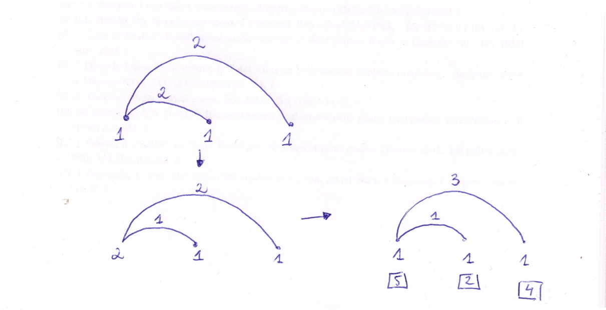
It's time for some theory

Theorem (Kalkowski, 2009)

Every graph has a total cool decoration with vertices decorated by the set $\{1,2\}$ *and edges decorated by the set* $\{1,2,3\}$.

Proof.

We can use the greedy approach to put proper values on each vertex and edge respectively.



1-2-3 Conjecture - versions cd.

Problems (list)				
name	type	S function	statement	status
edge (list 1-2-3)	C4	$S(v) = \sum_{x \in N(v)} f(xv)$	G with $\lvert E \rvert > 1$ has decoration from arbitrary lists of size 3.	open
total (list 1-2)	C5	$S(v) = f(v) + \sum_{x \in N(v)} f(vx)$	G has decoration from any lists of size 2.	open
weaker total (list 1-2)	T3		G has decoration from any lists of size 2 for V and any lists of size 3 for E.	proven by Wong and Zhu (2016)

It's time for some theory

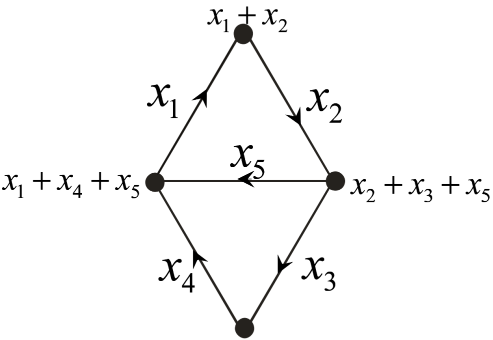
Surprisingly, there is a connection between these coloring problems and *Combinatorial Nullstellensatz*!

Theorem (Combinatorial Nullstellensatz)

Let P be a polynomial in $F[x_1, x_2, ..., x_m]$ over any field F. Suppose that there is a non-vanishing monomial in P whose degree is equal to the degree of P. Then, for arbitrary sets $A_i \subseteq F$, with $|A_i| = k_i + 1$, there is a choice of elements $a_i \in A_i$ such that $P(a_1, a_2, ..., a_m) \neq 0$.

How to use it?

Assign a variable x_e to each edge e of a graph G, and consider a polynomial $P = \prod_{uv \in E(G)}(S(u) - S(v))$, where $S(v)$ is the sum of variables assigned to the edges incident to the vertex v. We consider Clearly, any substitution for variables xe from lists $L(e) \in \mathbb{R}$ giving a non-zero value of P is a cool decoration of G. Thus, Conjecture 4 will follow if we could prove that P does not vanish over all posibilities.



$$
\begin{aligned}
P_G(x_1, \ldots, x_5) = & \\
&= (x_2 - x_4 - x_5)(x_3 + x_5 - x_1)(x_4 - x_2 - x_5) \times \\
&\times (x_1 + x_5 - x_3)(x_1 + x_4 - x_2 - x_3).
\end{aligned}
$$

It's time for some theory

A we see, *Combinatorial Nullstellensatz* approach led researchers to some interesting observations and even results! We'll prove one of them.

Theorem (Czerwiński, Grytczuk, Żelazny)

Every **planar bipartite** *graph has a cool* **vertex** *decoration from any lists of size three assigned to the vertices.*

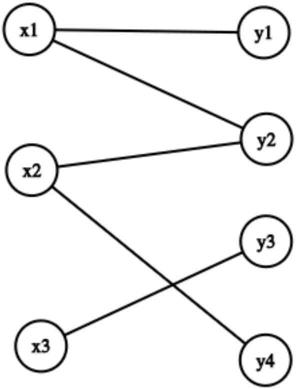
Proof.

Consider a polynomial

$$P = \prod_{uv \in E(G), u \in X, v \in Y} (S(u) - S(v)),$$

with variables x_u and y_v for X and Y respectively. □

Proof cd.



Proof cd.

Observations

- All variables for X appear with minus sign in the sum (and for Y with plus)
- none of monomials formed by choosing one variable from each factor $(S(u) - S(v))$ will eventually vanish in P
- A planar bipartite graph on n vertices can have at most $2n - 4$ edges (easy part), and therefore it can be oriented so that each vertex has at most two incoming edges Inot so easy)

Known result of Alon and Tarsi on 3-choosability of planar bipartite graphs use similar argument

Other extensions of these problems

Ironic decorations

Suppose that each vertex v of a graph G is assigned a real number f(v). Let $M(v) = f(v)d_v$ be the product of the assigned number by the degree d_v of the vertex v. We say that f is an ironic decoration of G if $M(u) \neq M(v)$ for every pair of adjacent vertices in G.

The question is if...

Conjecture

Every graph G has an ironic decoration by the set $\{1, 2, ..., \chi(G)\}$.

Please note that the conjecture is trivially true for regular graphs (deg's are equal).
Further we can pose this question in terms of lists coloring etc... etc...

Questions

References

Jarosław Grytczuk
"From the 1-2-3 Conjecture to the Riemann Hypothesis"
March 5, 2020
arXiv:2003.02887

https://smp.uph.edu.pl/msn/38/14-18.pdf