Wegner's conjecture
Colouring the square of a planar graph

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Introduction - Graphs

- By graph we mean a simple graph.
- $V(G), E(G)$
- The length of a path between two vertices is the number of edges on that path.
- We define the distance $dist_G(x, y)$ between two vertices x, y to be the length of the shortest path between them.
- $G^2 := (V, E^2)$, $E^2 := E \cup \{\{x, y\} : x, y \in V, dist_G(x, y) = 2\}$.
- $d_G(v)$ - degree of vertex v in graph G.
- $\Delta = \Delta(G)$ - the maximum degree of graph G.

Introduction - Coloring

- A vertex k-coloring of a graph G is a mapping $C : V \longrightarrow \{1, \cdots, k\}$ such that any two adjacent vertices u and v are mapped to different integers.
- The minimum k for which a coloring exists is called the chromatic number of G and is denoted by $\chi(G)$.

The Four Color Theorem

The Four Color Theorem

For every planar graph G: $\chi(G) \leqslant 4$.

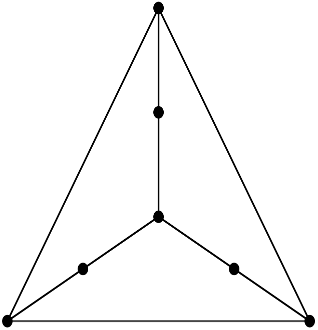
Wegner's conjecture

Wegner's conjecture, 1977

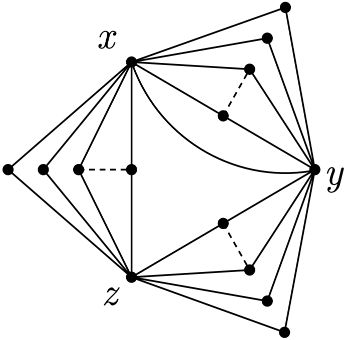
Let G be a planar graph with maximum degree Δ. Then

$$\chi(G^2) \leqslant \begin{cases} 7, & \text{if } \Delta \leqslant 3 \\ \Delta + 5, & \text{if } 4 \leqslant \Delta \leqslant 7 \\ \left\lfloor \frac{3\Delta}{2} \right\rfloor + 1, & \text{if } \Delta \geqslant 8 \end{cases}$$

These bounds are best possible



graph G, $\Delta \leqslant 3$



graph H, even $\Delta \geqslant 8$

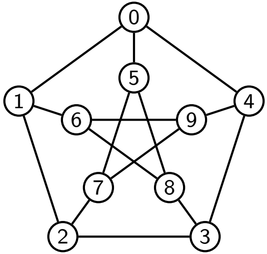
Important observation

General case

$\Delta + 1 \leqslant \chi(G^2) \leqslant \Delta^2 + 1.$

Planar graphs

Wegner's conjecture implies that $\chi(G^2) = O(\Delta)$.



Easy greedy algorithms don't work

- Every planar graph has a vertex with degree at most 5.
- $G = \{v_1, v_2, \cdots, v_n\}$, such that v_i has at most 5 neighbours in $\{v_1, v_2, \cdots, v_{i-1}\}$.
- Unfortunately, v_i has at most 5 neighbours in $\{v_1, v_2, \cdots, v_{i-1}\}$ DOES NOT IMPLY that v_i has at most $5(\Delta - 1)$ vertices at distance two in $\{v_1, v_2, \cdots, v_{i-1}\}$.

First linear in Δ upper bound

Theorem - Jonas, 1993

If G is a planar graph with maximum degree $\Delta \geqslant 5$, then $\chi(G^2) \leqslant 9\Delta - 19$.

Proof (idea):

- order the vertices $\{v_1, \cdots, v_n\}$ in such a way that each v_i has at most 5 neighbours on the left.
- greedily assign colors in that order.
- we must show that every vertex v_i has at most $9\Delta - 20$ vertices at distance one or two in G on the left.
- assume v_i has $0 \leqslant k \leqslant 5$ neighbours on the left.

First linear in Δ upper bound

- assume that there is a path $v_i ba$, $a \in \{v_1, \cdots, v_{i-1}\}$
- we have two cases:
- first case: $b \in \{v_1, \cdots, v_{i-1}\}$, then we have at most $k(\Delta - 1)$ such paths $v_i ba$.
- second case: $b \notin \{v_1, \cdots, v_{i-1}\}$, then there are at most $\Delta - k$ of such b. Also since b has at most 5 neighbours on his left, and one of those neighbours is v_i, b can have at most 4 neighbours on the left of v_i. So in this case we have at most $4(\Delta - k)$ such paths $v_i ba$.
- finally we find at most $k + k(\Delta - 1) + 4(\Delta - k)$ vertices at distance on or two from v_i on the left of v_i.
- for $\Delta \geqslant 5$ and $0 \leqslant k \leqslant 5$, this number is at most $9\Delta - 20$.

What we know until now? - general case

- T. Jonas, 1993 - $\chi(G^2) \leqslant 9\Delta - 19$, if $\Delta \geqslant 5$.
- T. Jonas, 1993 - $\chi(G^2) \leqslant 8\Delta - 22$, if $\Delta \geqslant 7$.
- S. Wong, 1996 - $\chi(G^2) \leqslant 3\Delta + 5$, if $\Delta \geqslant 7$.
- T. Madaras, A. Marcinova, 2002 - $\chi(G^2) \leqslant 2\Delta + 8$, if $\Delta \geqslant 12$.
- J. van den Heuvel, S. McGuiness, 2003 - $\chi(G^2) \leqslant 2\Delta + 25$.
- Agnarsson, Halldorsson, 2003 - $\chi(G^2) \leqslant \left\lfloor \frac{9\Delta}{5} \right\rfloor + 2$, if $\Delta \geqslant 749$.
- Molloy, Salavatipour, 2005 - $\chi(G^2) \leqslant \left\lfloor \frac{5\Delta}{3} \right\rfloor + 78$.

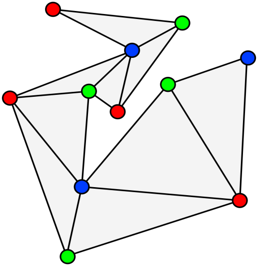
What we know until now? - special cases

S. Hartke - 2016, C. Thomassen - 2018

If G is a subcubic planar graph, then $\chi(G^2) \leqslant 7$.

What we know until now? - special cases

A planar graph is said to be outerplanar if it has a plane embedding such that all vertices lie on the boundary of the unbounded face.



Theorem - K. Lih, W. Wang, 2006

If G is an outerplanar graph with maximum degree $\Delta \geqslant 3$, then $\chi(G^2) \leqslant \Delta + 2$. Moreover, $\chi(G^2) = \Delta + 1$, if $\Delta \geqslant 7$.

What we know until now? - bounded girth

The girth of a graph is the length of a shortest cycle contained in the graph.

Theorem 1 - Cranston, Kim

If G is planar, $\Delta(G) = 3$, and *girth* $\geqslant 7$, then $\chi(G^2) \leqslant 7$.

Theorem 2 - Cranston, Kim

If G is planar, $\Delta(G) = 3$, and *girth* $\geqslant 9$, then $\chi(G^2) \leqslant 6$.

What we know until now? - bounded $mad(G)$

Maximal average degree of a graph G

$$mad(G) := \max_{H \subset G} \frac{2|E(H)|}{|V(H)|}$$

Theorem - Cranston, Erman

Let G be a graph with maximum degree $\Delta = 4$. Then if $mad(G) < \frac{16}{7}, \frac{22}{9}, \frac{18}{7}, \frac{14}{5}$, respectively, then G^2 is $5, 6, 7, 8$-colorable, respectively.

Asymptotically true!

Theorem - Havet, Heuvel, Reed

The square of every planar graph G of maximum degree Δ has chromatic number at most $(1 + o(1))\frac{3}{2}\Delta$.

This is the end!
Thank you!