Codes from zero-divisor graphs

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Combinatorial Optimization Seminar

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work under progress

Joint work under progress in collaboration with Khalid Abdelmoumen, Driss Bennis and Fouad Taraza.

Plan

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A linear (binary) code is a linear subspace C of the vector space $(\mathbb{F}_2)^n$, where $\mathbb{F}_2$ is the finite field with 2 elements and n is a positive integer.

- The integer n is called the length of C.
- The dimension of C is its rank.
- An element of a linear code is called a codeword.

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Construction

To construct a linear code, we use the so called generator matrix as follows :

Let C be a linear code of length n and rank k.

There is a matrix G of type $k \times n$ such that

$$v \in C \Longleftrightarrow \exists v' \in (\mathbb{F}_2)^k, v'G = v.$$

In other words, $C = Im(G)$.

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The Hamming distance or simply the distance between two codewords m_1 and m_2, denoted by $d(m_1, m_2)$, is the number of elements in which they differ.

- The weight of a codeword m, denoted by $wt(m)$, is the number of nonzero coordinates in m.
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The minimal distance of a linear code C, denoted by $d(C)$, is the minimum distance between distinct codewords.

Example

Consider the linaear code

$C = \{(0,0,0,0),(1,1,1,1)(0,1,0,1)(1,0,1,0)\}$, *is of :*

- *length* 4.
- *rank* 2.
- *minimal distance* 2.

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Correction and detection of errors

Theorem

A linear code of minimum distance d can detect $d-1$ errors and can correct up to $t=[\frac{d-1}{2}]$ errors.

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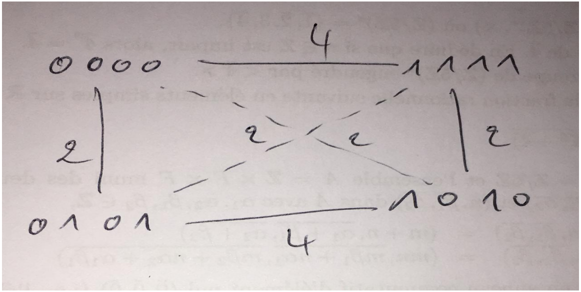

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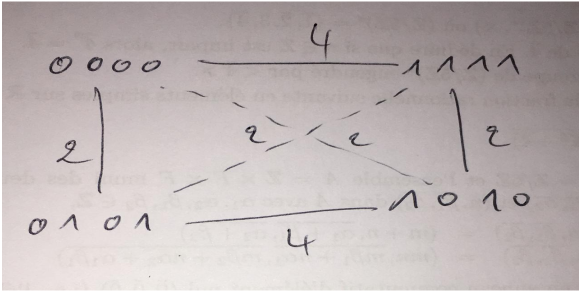

P. Dankelmann, J. D. Key and B. G. Rodrigues. Codes from incidence matrices of graphs. Designs, Des. Codes Cryptogr. **68** :373–393, 2013.

Codes from graph

An edge-cut of a connected graph Γ is a set $S \subset E$ such that $\Gamma \setminus S = (V, E \setminus S)$ is disconnected. The edge-connectivity $\lambda(\Gamma)$ is the cardinality of the minimal edge-cut.

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Let Γ be a connected graph. We denote by $\delta(\Gamma)$, the minimum degree of the vertices of Γ.

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Zero-divisor graphs

Definition (Anderson and Livingston, 1998)

The zero-divisor graph of ring R denoted by $\Gamma(R)$, is the simple graph whose set of vertices consists of all nonzero zero divisors of R such that two distinct vertices x and y are joined by an edge if $xy = 0$.

It is well-known that the zero-divisor graph of a commutative ring has the following properties :

- It is a connected graph.
- Its diameter is at most 3.
- When it contains a cycle, its girth is 3 or 4.

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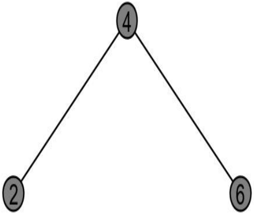
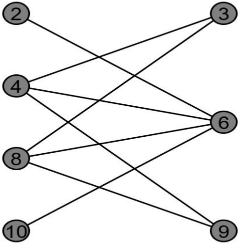


$\Gamma(\mathbb{Z}_8)$



$\Gamma(\mathbb{Z}_{12})$

$\Gamma(\mathbb{Z}_n)$ are super lambda graphs

Theorem

Let n be a nonzero integer such that $n = p_1^{\alpha_1} \dots p_m^{\alpha_m}$ with p_i is a prime integer and $\alpha_i \geq 1$ for all $i \in \{1, ..., m\}$. Then, $\Gamma(\mathbb{Z}_n)$ is a super λ graph with $\lambda = p_r - 1$ such that $p_r = min\{p_i | i \in \{1, ..., m\}\}$.

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- *length* $\frac{1}{2}\sum_{x \in Z(\mathbb{Z}_n)^*} |Ann(x) \setminus \{x\}|$,
- *rank* $|Z(\mathbb{Z}_n)^*| - 1$,
- *minimum Hamming distance* $p_r - 1$ *with* $p_r = min\{p_i | i \in \{1, ..., m\}\}$.

and the codewords of the minimum weight are the rows of G_n of weight $\delta(\Gamma)$.

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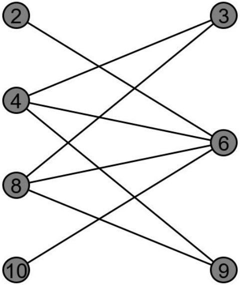

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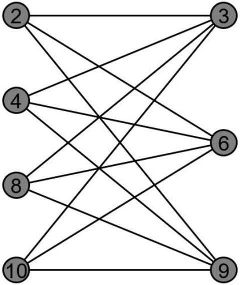

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Let p be a prime integer and $\alpha > 2$ an integer. The binary code $C_2(\overline{G}_{p^\alpha})$ is of :

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p	α	parameters of $C_2(G_{p^\alpha})$: $[n, k, d]$	parameters of $C_2(\overline{G}_{p^\alpha})$: $[n, k, d]$
2	3	[2,2,1]	[3,2,2]
2	4	[7,6,1]	[21,6,6]
2	5	[23,14,1]	[105,14,14]
2	6	[61,30,1]	[465,30,30]
3	3	[13,7,2]	[28,7,7]
3	4	[64,25,2]	[325,25,25]
5	3	[86,23,4]	[276,23,23]

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