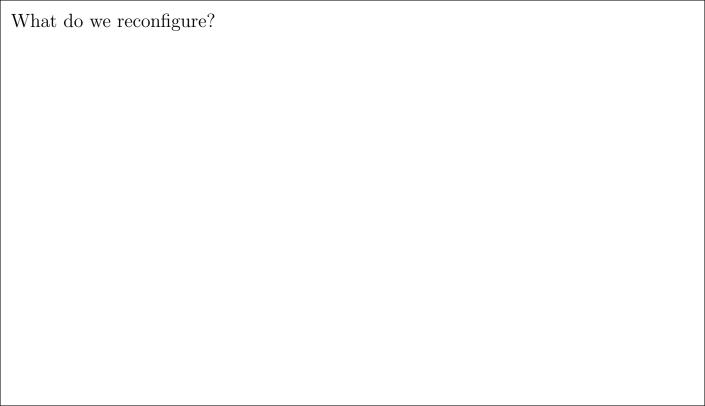
Reconfiguring Independent Sets On Interval Graphs

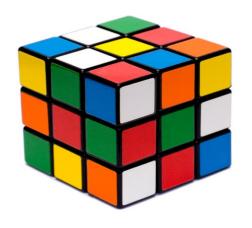
Marcin Briański, Stefan Felsner, Jędrzej Hodor and Piotr Micek



What do we reconfigure?

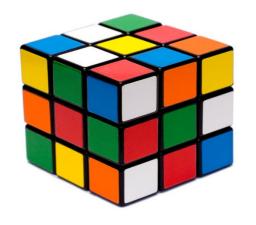
What do we reconfigure?

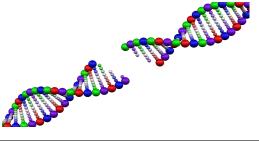


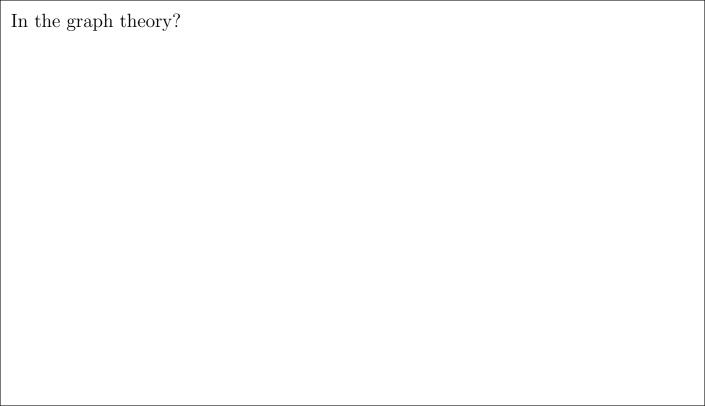


What do we reconfigure?

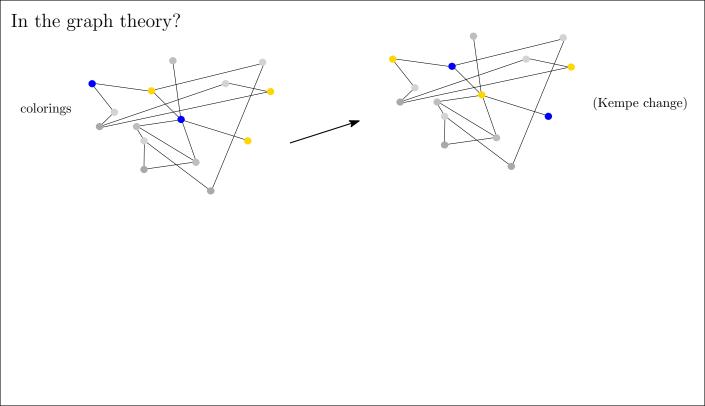


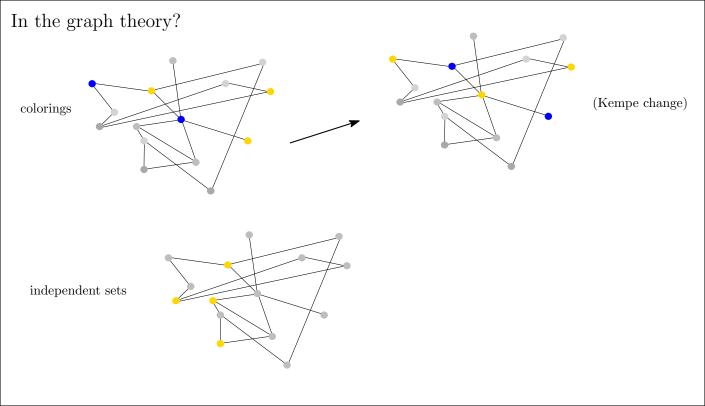


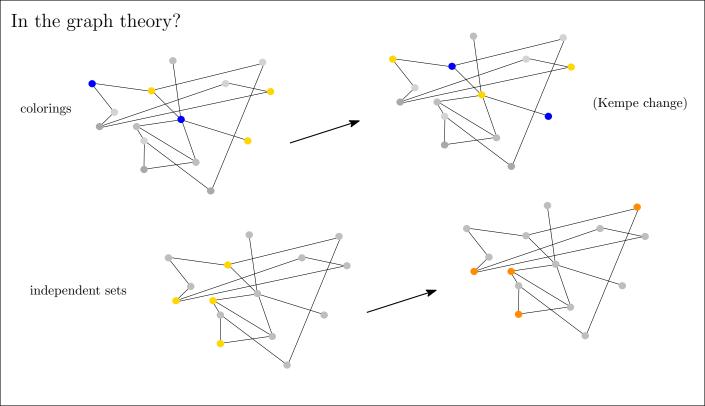




In the graph theory? colorings







Reconfiguration graph			
	V(G)	E(G)	
		'	

Reconfiguration graph	ı	
V(G)	E(G)	
configurations	configurability	
	I	

Reconfiguration graph		
V(G)	E(G)	
configurations	configurability	
Rubic's cube configuration cars configuration DNA sequence	one car is moved straigth difference by one nucleotide	

V(G)	E(G)
configurations	configurability
Rubic's cube configuration cars configuration DNA sequence coloring	one side moved one car is moved straigth difference by one nucleotide Kempe change

Reconfiguration graph

V(G)	E(G)
configurations	configurability
Rubic's cube configuration cars configuration	one side moved one car is moved straigth
DNA sequence	difference by one nucleotide

Kempe change

 $I\ominus J=\{u,v\}$ and $\{u,v\}\in E$

Reconfiguration graph

coloring

independent set

$\frac{V(G)}{\text{configurations}}$

cars configuration

DNA sequence

independent set

coloring

Rubic's cube configuration

Reconfiguration graph

configurability

one side moved

Kempe change

one car is moved straigth

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V(G)	E(G)
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	1

 $I,J\in V(G)$

Reconfiguration graph

Questions?

V(G)	E(G)
configurations	configurability
Rubic's cube configuration cars configuration	one side moved one car is moved straigth
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Reconfiguration graph

coloring

independent set

Questions?

 $I,J \in V(G)$

 $I\ominus J=\{u,v\}$ and $\{u,v\}\in E$

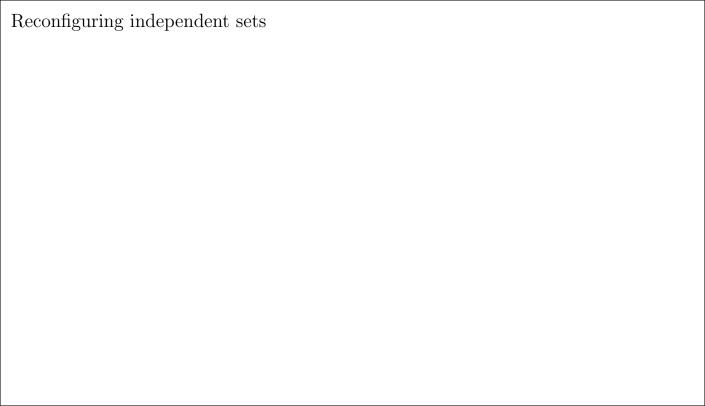
• Does there exist a path from I to J in G?

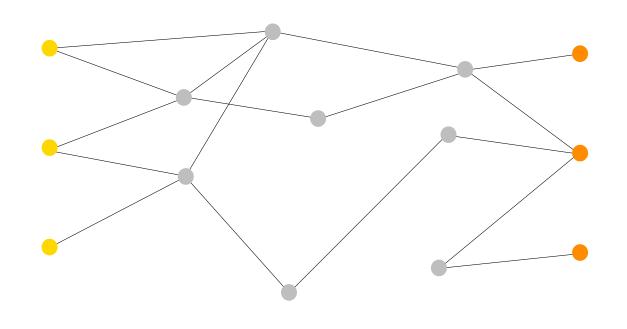
Kempe change

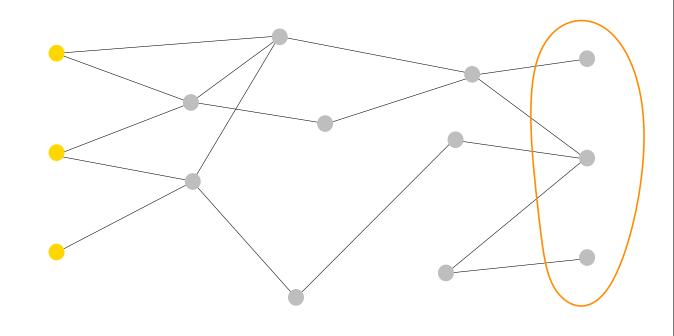
Reconfiguration graph		
V(G)	E(G)	
configurations	configurability	
Rubic's cube configuration cars configuration DNA sequence coloring independent set	one side moved one car is moved straigth difference by one nucleotide Kempe change $I\ominus J=\{u,v\} \text{ and } \{u,v\}\in E$	
Questions?	$I,J\in V(G)$	
	• Does there exist a path from I to J in G?	
	• What is the diameter of G ?	

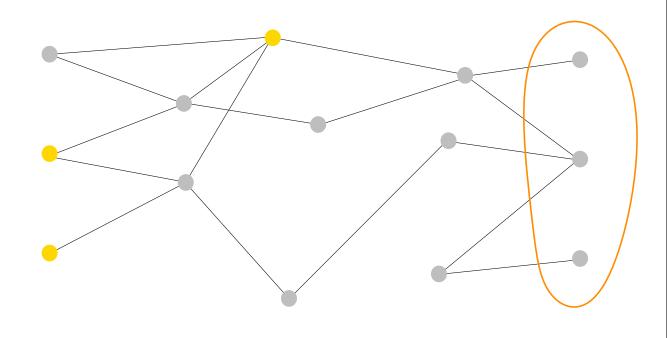
Reconfiguration graph		
V(G)	E(G)	
configurations	configurability	
Rubic's cube configuration cars configuration DNA sequence coloring independent set	one side moved one car is moved straigth difference by one nucleotide $ \text{Kempe change} $ $ I\ominus J=\{u,v\} \text{ and } \{u,v\}\in E $	
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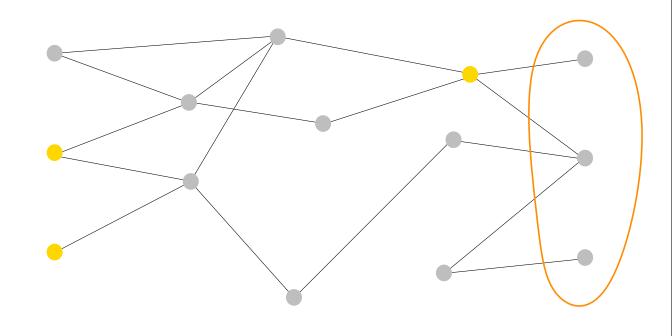
Reconfiguration graph V(G)E(G)configurations configurability Rubic's cube configuration one side moved cars configuration one car is moved straigth difference by one nucleotide DNA sequence coloring Kempe change $I \ominus J = \{u, v\}$ and $\{u, v\} \in E$ independent set Questions? $I, J \in V(G)$ • Does there exist a path from I to J in G? • What is the diameter of G? • Is G connected? TS-Reachability

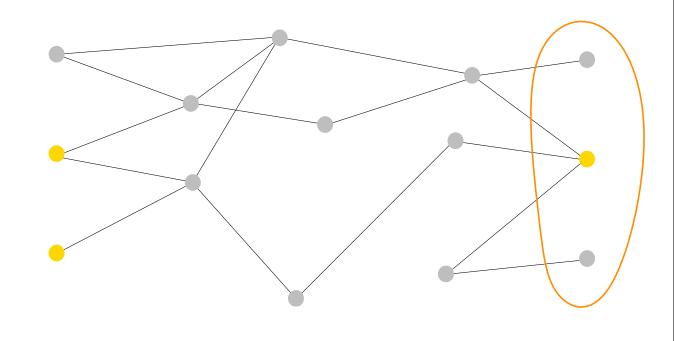


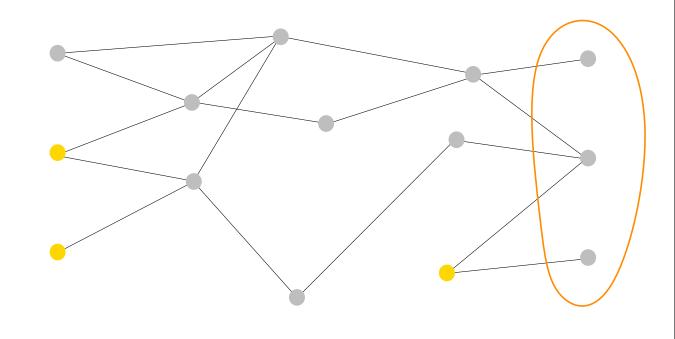


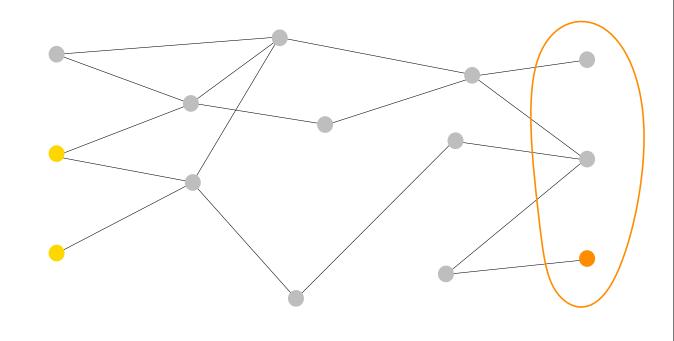


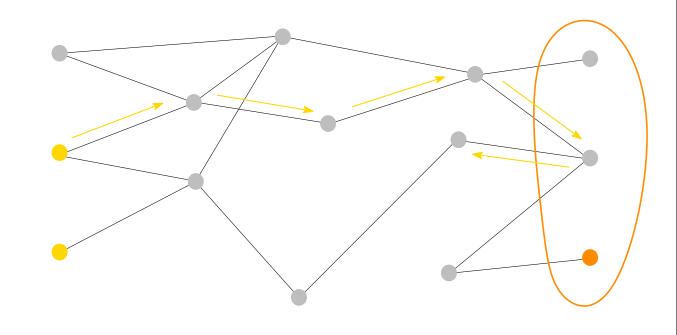


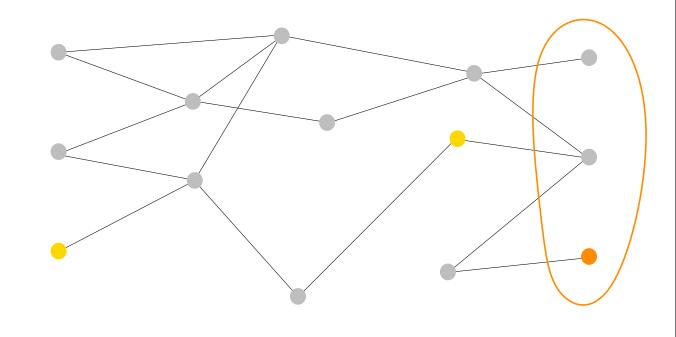


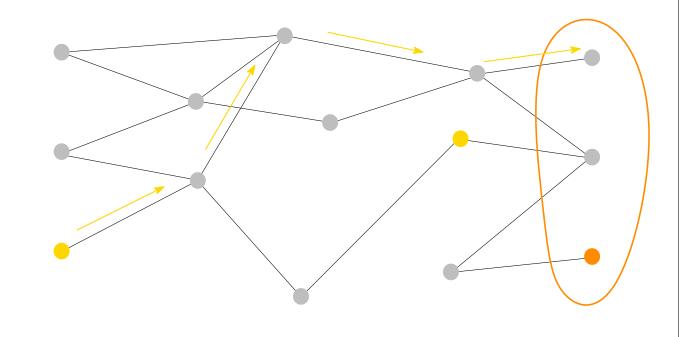


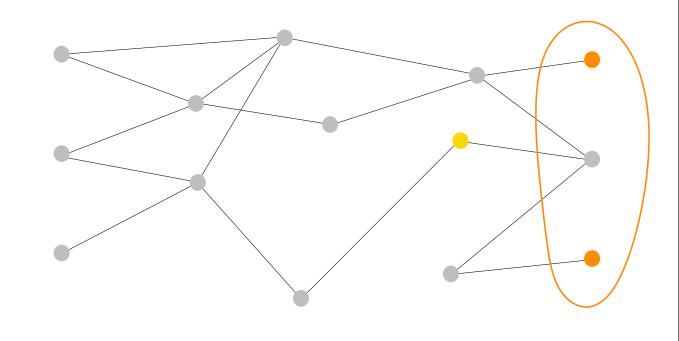


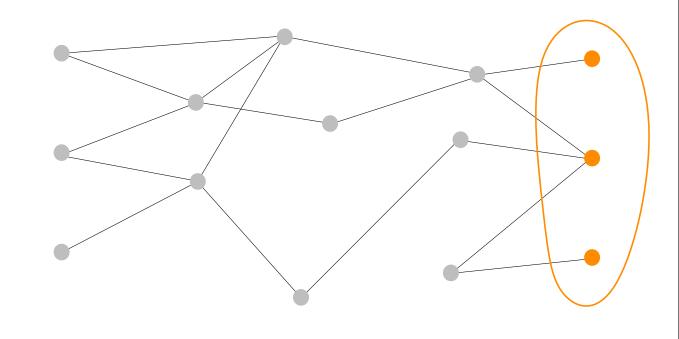


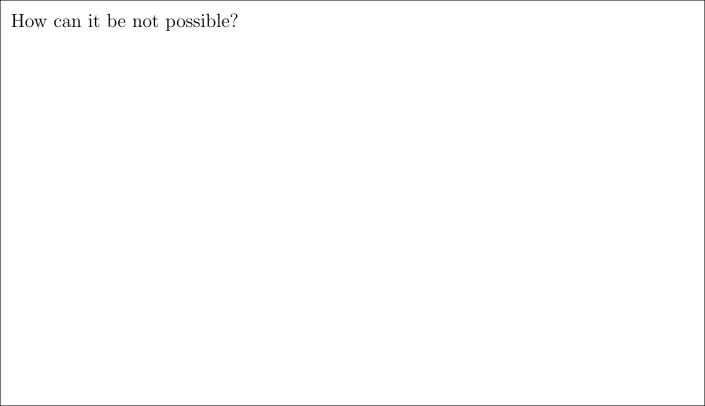


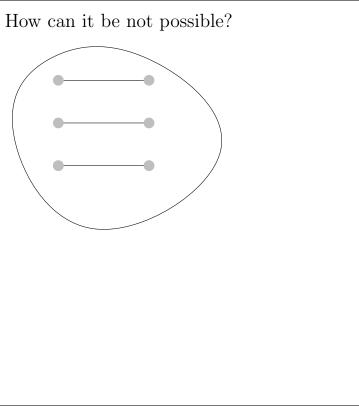


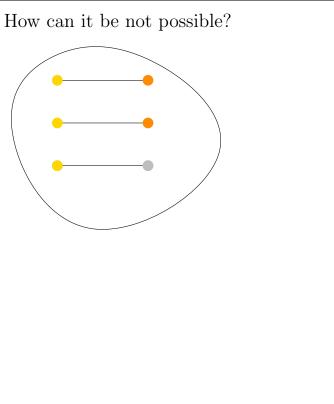


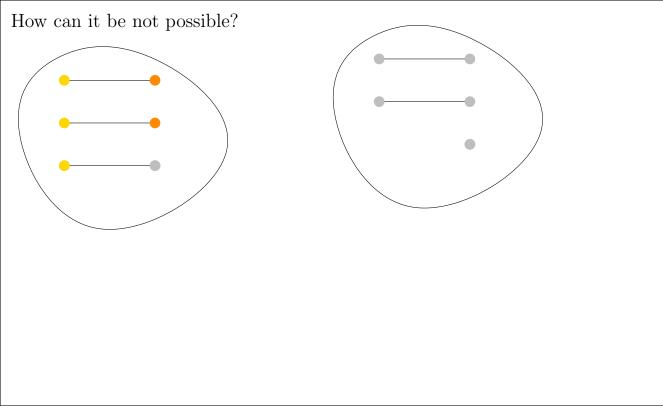


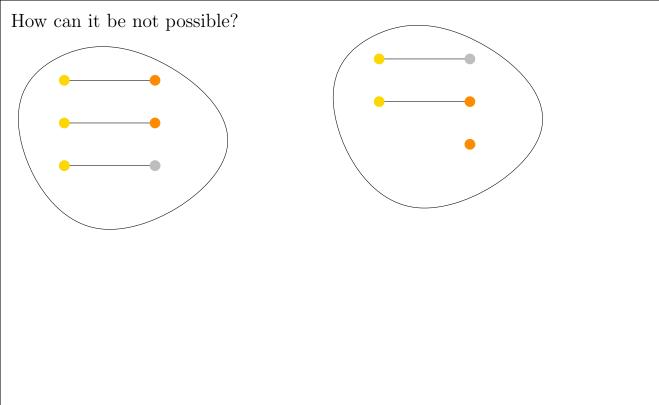


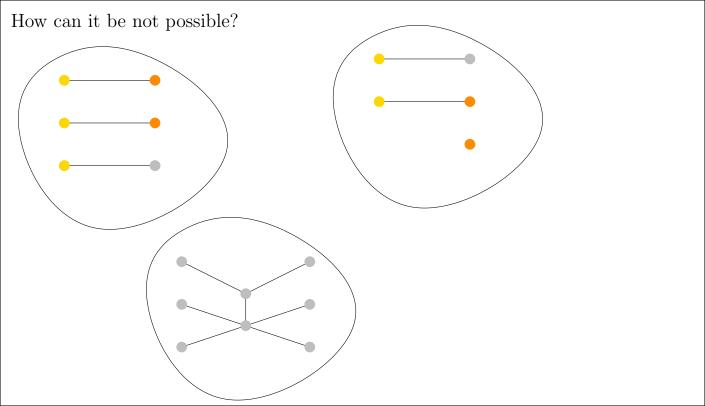


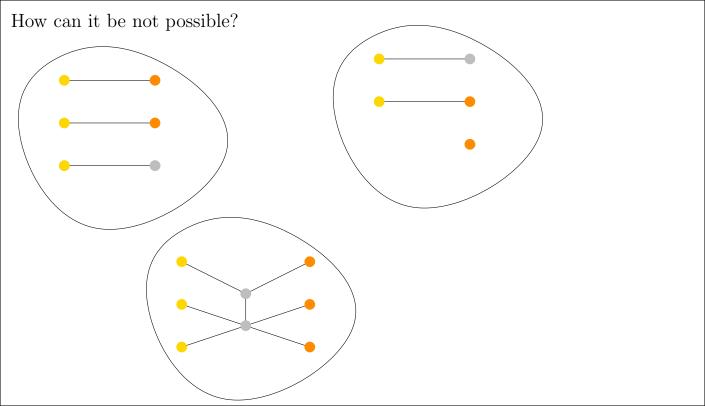


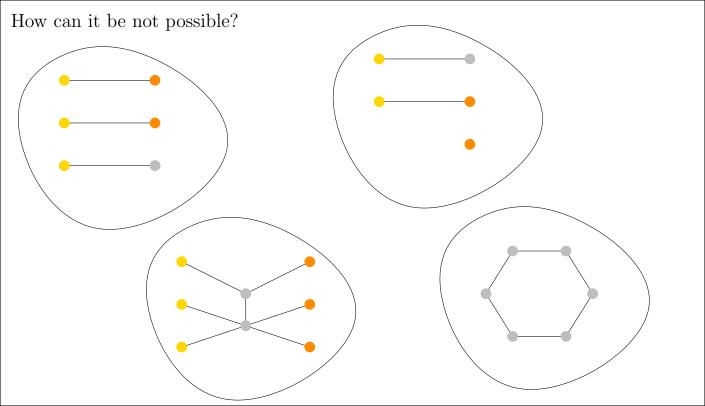


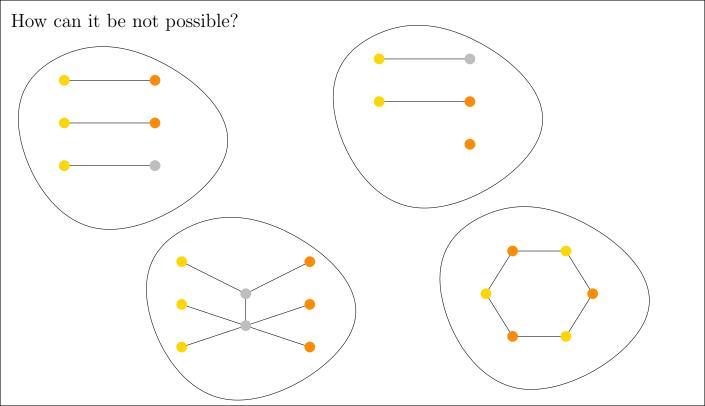


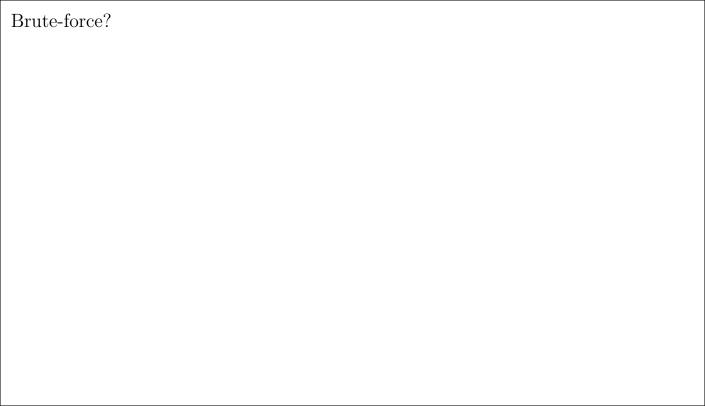










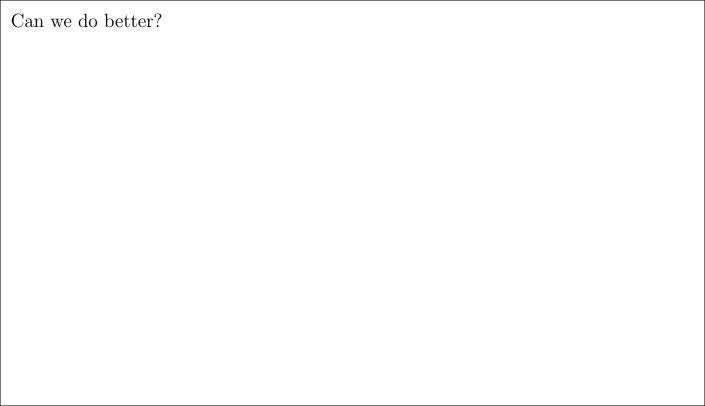


Brute-force?
$G-\operatorname{graph}$
V(G) = n
k – size of independent sets
$R_k(G)$ – reconfiguration graph

Brute-force?	
G - graph V(G) = n k - size of independent sets	$ V(R_k(G)) \leq \binom{n}{k} \sim \Theta\left(n^k\right)$
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Brute-force?	
G - graph $ V(G) = n$	$ V(R_k(G)) \leq \binom{n}{k} \sim \Theta\left(n^k\right)$
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Brute-force?	
G - graph V(G) = n k - size of independent sets $R_k(G) - \text{reconfiguration graph}$	$ V(R_k(G)) \leqslant \binom{n}{k} \sim \Theta\left(n^k\right)$ DFS? Dijkstra?



Can we do better? In: G graph, I, J independent sets in G Out: Is there a path from I to J in $R_{|I|}(G)$?

TS-Reachability	I independent sets in G th from I to J in $R_{ I }(G)$?
Hard	Open
	Out: Is there a pa

Can	we	${\rm do}$	better?

TS-Reachability

Out: Is there a path from I to J in $R_{|I|}(G)$?

In: G graph, I, J independent sets in G

Polynomial	Hard	Open
tree ['11]		
cograph ['12]		
claw-free ['14]		
bipartite permutation ['15]		
bipartite distance heridiatary ['15]		
interval [Bonamy, Bousquet '18]		

TS-Reachability

In: G graph, I, J independent sets in GOut: Is there a path from I to J in $R_{|I|}(G)$?

Polynomial	Hard	Open	
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interval [Bonamy, Bousquet '18]	incomparability ['21]		

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TS-Reachability

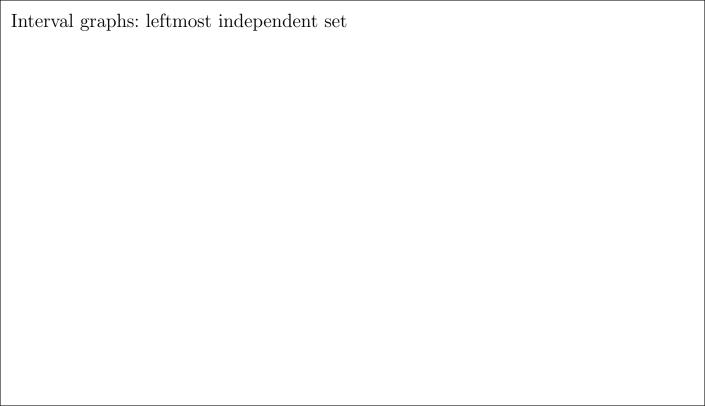
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dynamic algorithmno bound on the path length			

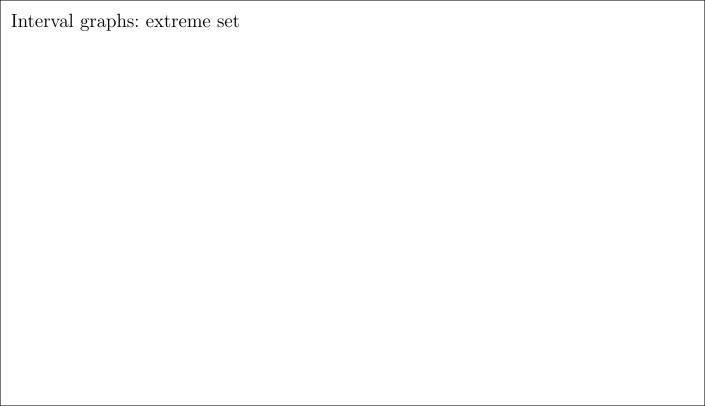


O P	leftmost ind	- F		
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	_			

Interval graphs: leftmost i	ndependent set	
1 0		
k = 3		

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<u> </u>
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ullet if two independent sets can reach the leftmost, then concatenate paths

Interval graphs: leftmost independent set
k = 3
if two independent sets can reach the leftmost, then concatenate pathswhat if not?



Interval graphs: extreme set				
${\mathcal C}$ - connected component of $R_k(G)$				

Interval graphs: extreme set \mathcal{C} - connected component of $R_k(G)$ I - independent set, then $I = \{I_1, I_2, \dots, I_k\}$ in the natural order Interval graphs: extreme set $\mathcal{C} \text{ - connected component of } R_k(G)$ $I \text{ - independent set, then } I = \{I_1, I_2, \dots, I_k\} \text{ in the natural order}$

 $ex_i(\mathcal{C}) = min_r\{I_i \mid I \in \mathcal{C}\}\$

Interval graphs: extreme set \mathcal{C} - connected component of $R_k(G)$

 $\mathrm{EX}(\mathcal{C}) = \{\mathrm{ex}_1(\mathcal{C}), \mathrm{ex}_2(\mathcal{C}), \dots, \mathrm{ex}_k(\mathcal{C})\}$

I - independent set, then $I = \{I_1, I_2, \dots, I_k\}$ in the natural order

$$\operatorname{ex}_i(\mathcal{C}) = \min_r \{ I_i \mid I \in \mathcal{C} \}$$

Interval graphs: extreme set

$$\mathcal{C}$$
 - connected component of $R_k(G)$

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$$\mathrm{ex}_j(\mathcal{C}) = \mathrm{min}_r\{I_j \mid I \in \mathcal{C}\}$$

$$\operatorname{ex}_{j}(\mathcal{C}) = \min_{r} \{ I_{j} \mid I \in \mathcal{C} \}$$

$$EX(\mathcal{C}) = \{ex_1(\mathcal{C}), ex_2(\mathcal{C}), \dots, ex_k(\mathcal{C})\}\$$

LGO: for I finds
$$A(I) = EX(C)$$
 where $I \in C$

ALGO: for
$$I$$
 finds $A(I) = EX(\mathcal{C})$, where $I \in \mathcal{C}$

For
$$I, J$$
 check if $A(I) == A(J)$

Interval graphs: extreme set

$$\mathcal{C}$$
 - connected component of $R_k(G)$

I - independent set, then $I = \{I_1, I_2, \dots, I_k\}$ in the natural order

$$\operatorname{ev}_{\cdot}(\mathcal{C}) = \min \{I \mid I \in \mathcal{C}\}$$

 $ex_i(\mathcal{C}) = min_r\{I_i \mid I \in \mathcal{C}\}\$

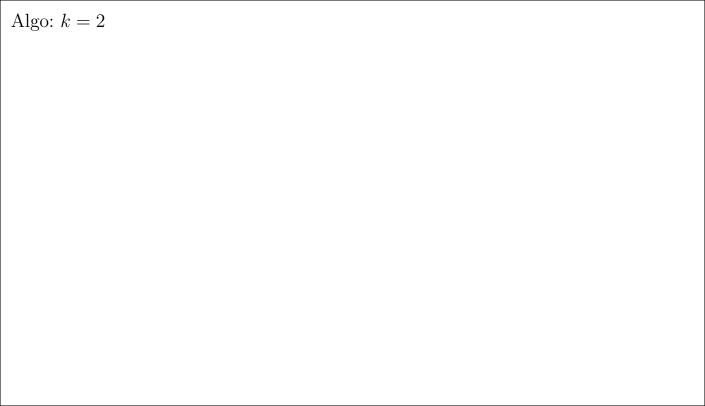
$$I_j(\mathcal{C}) = \min_r \{ I_j \mid I \in \mathcal{C} \}$$

 $\mathrm{EX}(\mathcal{C}) = \{\mathrm{ex}_1(\mathcal{C}), \mathrm{ex}_2(\mathcal{C}), \dots, \mathrm{ex}_k(\mathcal{C})\}$ "try to go left as far as you can"

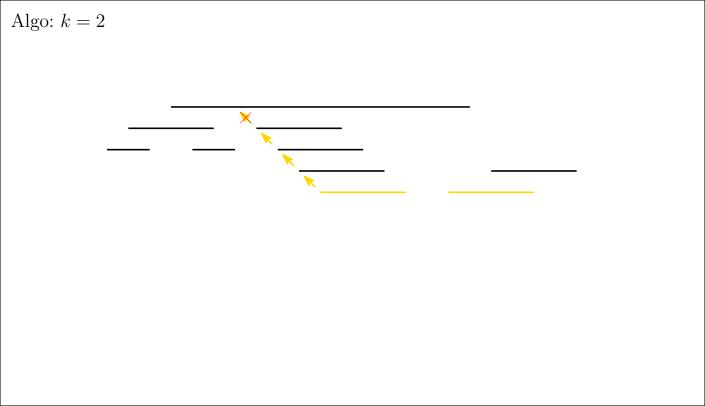
ALGO: for
$$I$$
 finds $A(I) = EX(\mathcal{C})$, where $I \in \mathcal{C}$

For I, J check if A(I) == A(J)

$$ck if A(I) == A(J)$$

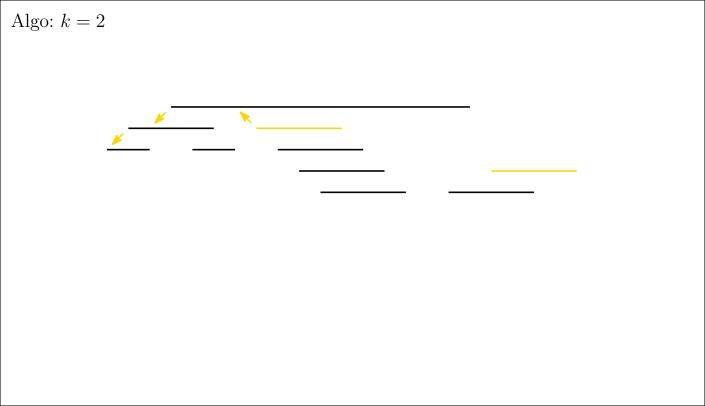


Algo: $k=2$		



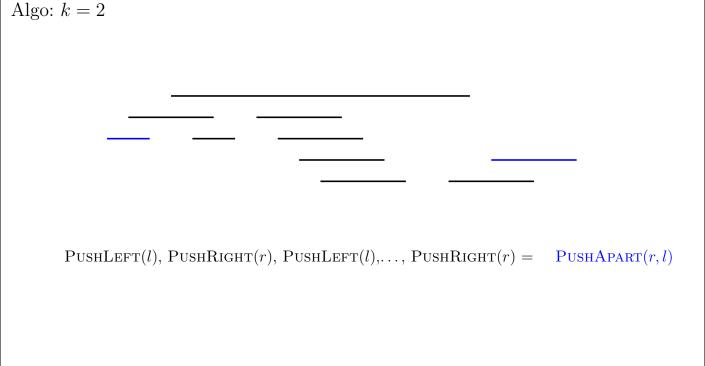
Algo: $k=2$		
	 	 -

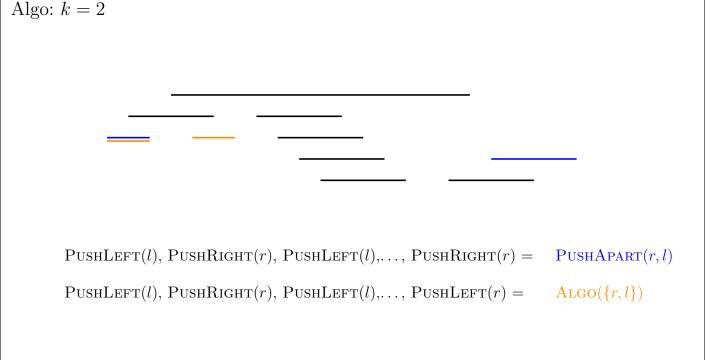
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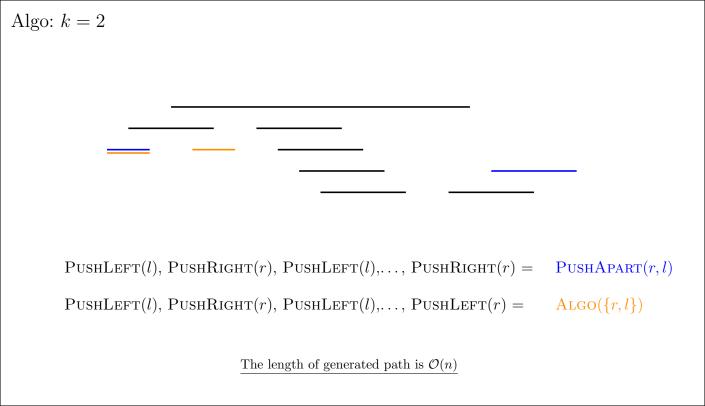


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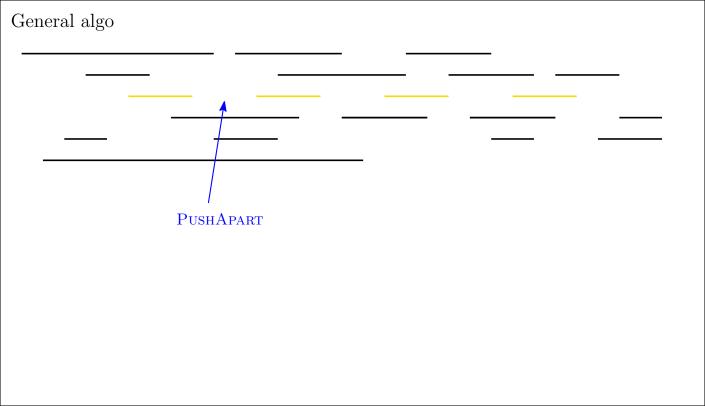
k = 2				
			-	
PushLe	рет (l) , PushRigh	T(r), PushLe	$ ext{CFT}(l), \dots$	

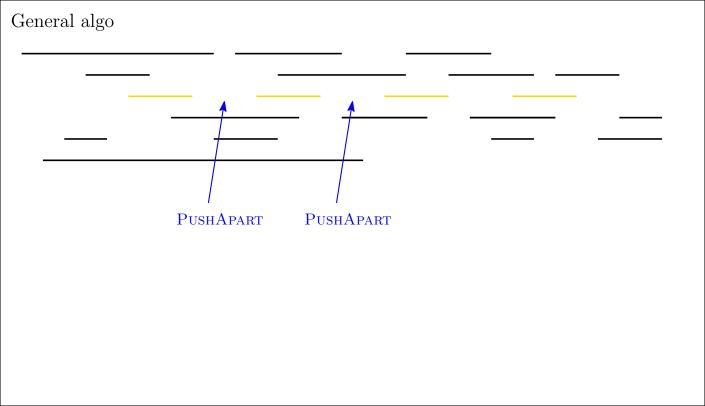


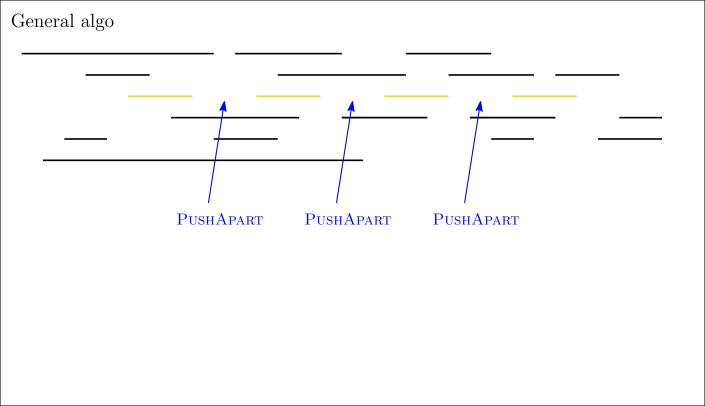


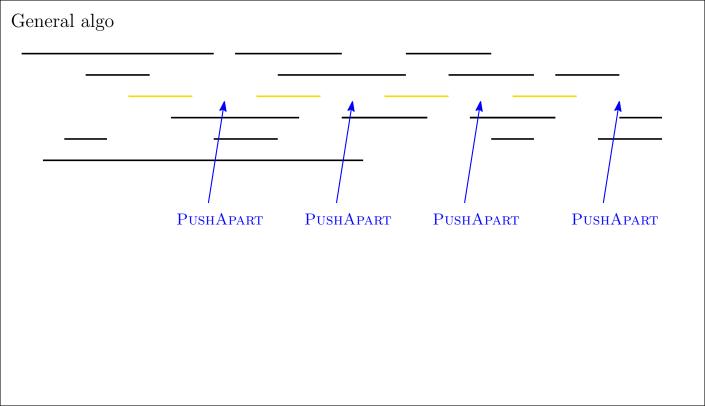


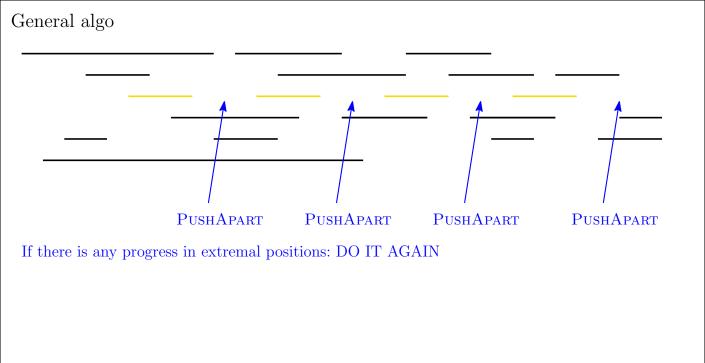
General algo			
	 	 _	_

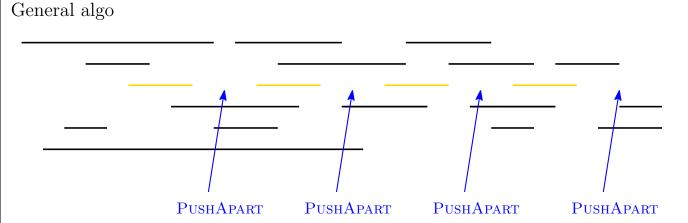




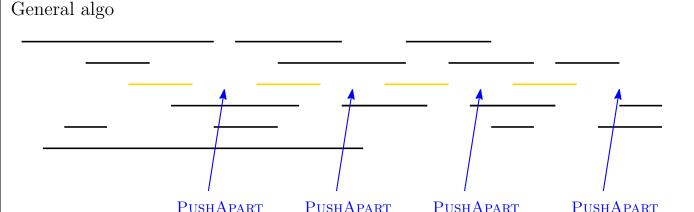




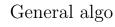


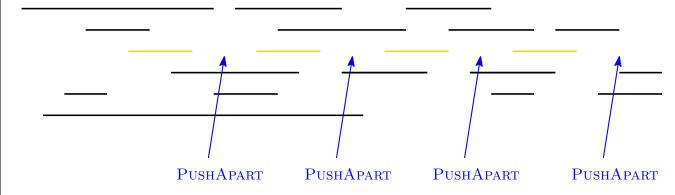


• Single PushApart makes $\mathcal{O}(n)$ operations



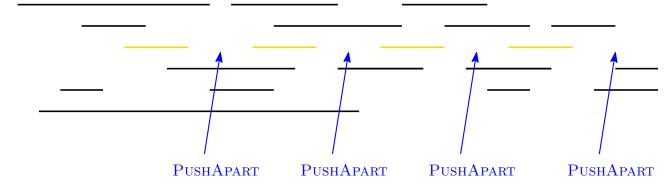
- Single PushApart makes $\mathcal{O}(n)$ operations
- \bullet Single round takes $\mathcal{O}(k\cdot n)$ operations





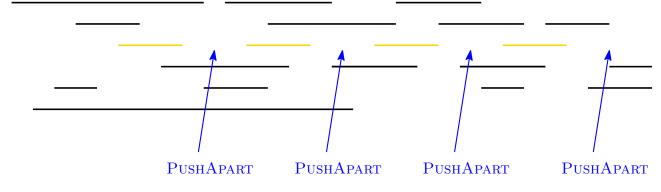
- Single PushApart makes $\mathcal{O}(n)$ operations
- Single round takes $\mathcal{O}(k \cdot n)$ operations
- \bullet For each token there are only $2 \cdot n$ possible extremal positions (left, right)

General algo



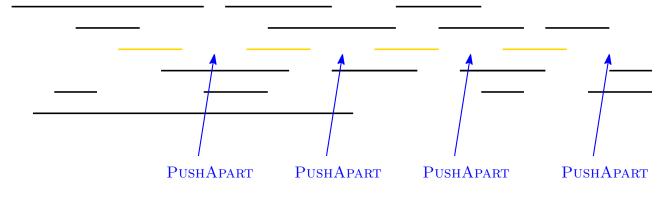
- \bullet Single PushApart makes $\mathcal{O}(n)$ operations
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$General\ algo$

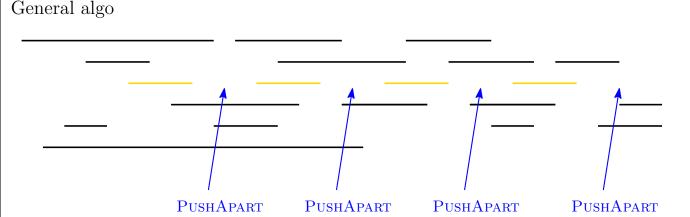


- \bullet Single PushApart makes $\mathcal{O}(n)$ operations
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- A progress can be made only $\mathcal{O}(k \cdot n)$ times
- This gives $\mathcal{O}(k^2n^2)$ operations

 $General\ algo$

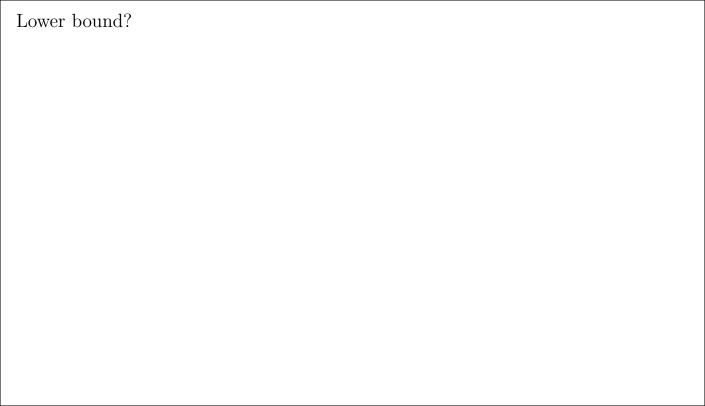


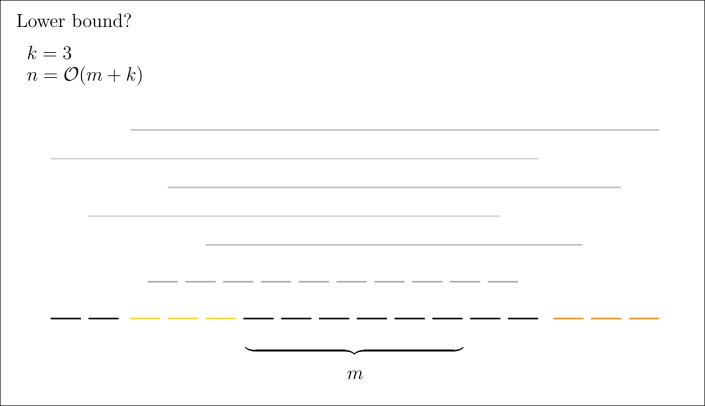
- Single PushApart makes $\mathcal{O}(n)$ operations • Single round takes $\mathcal{O}(k \cdot n)$ operations
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- Better order of PushApart and more careful analysis

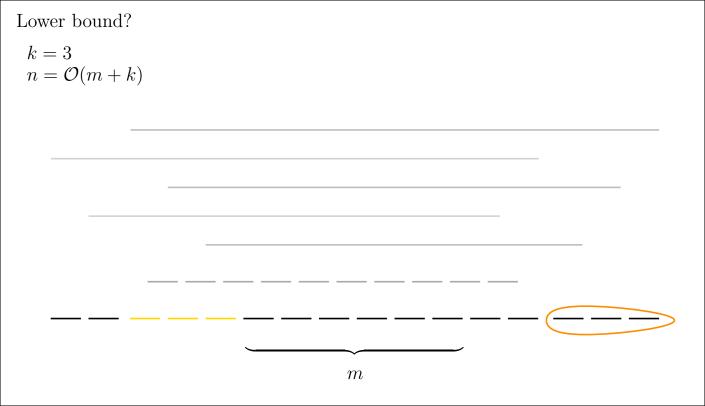


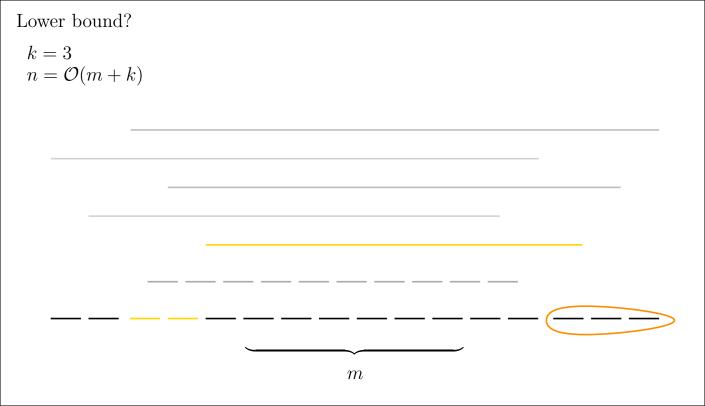
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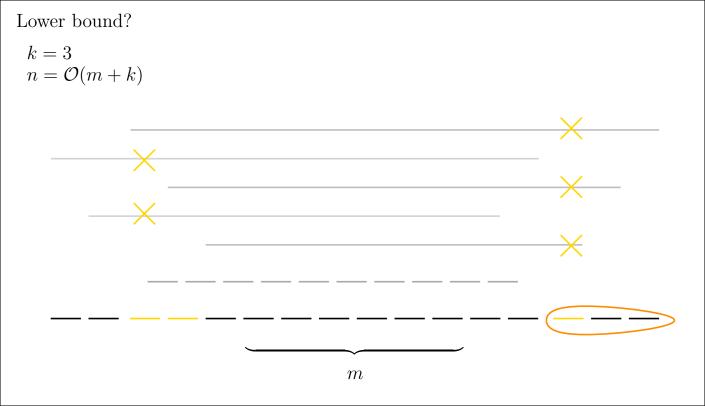
 Better order of PushApart and more careful analysis
- Bound $\mathcal{O}(kn^2)$ for the path length!

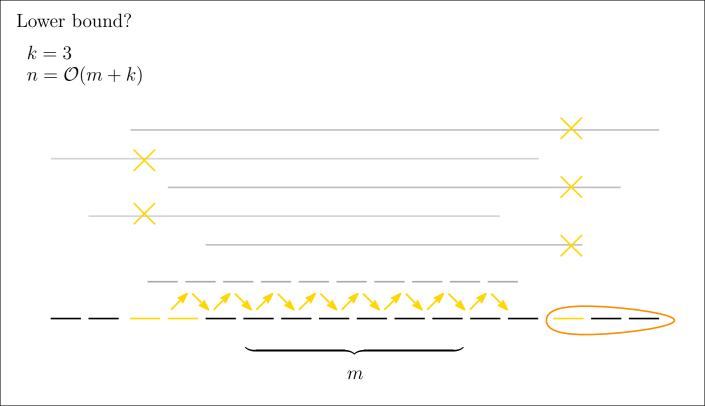


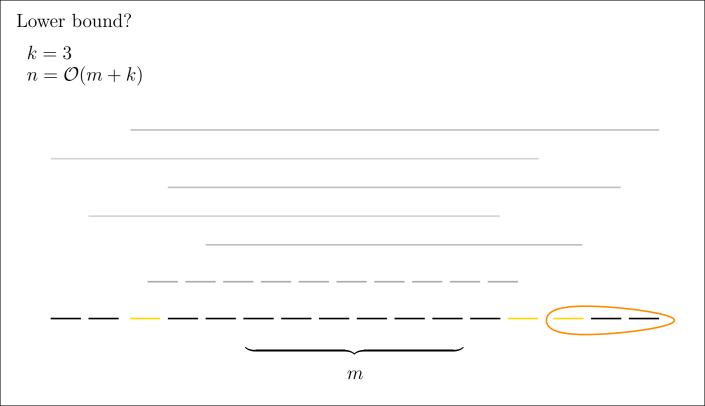


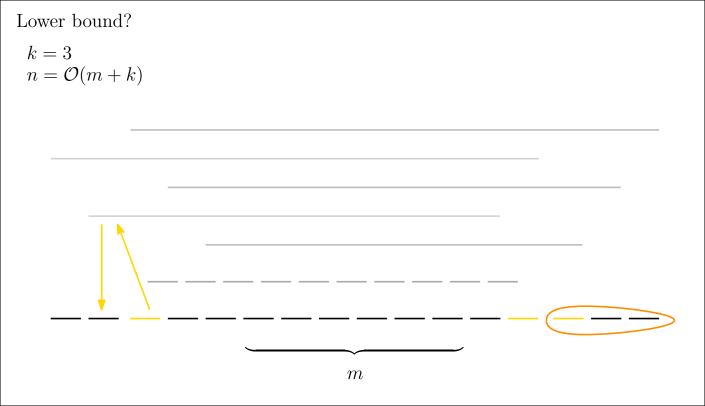


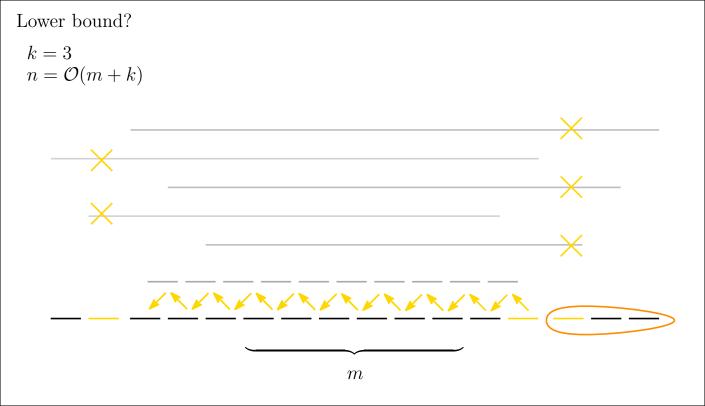


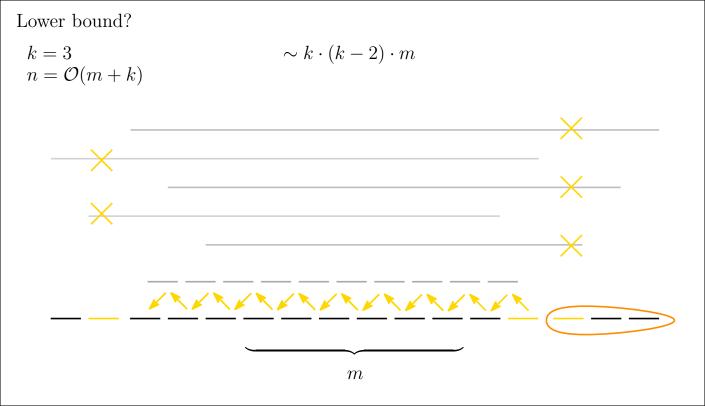


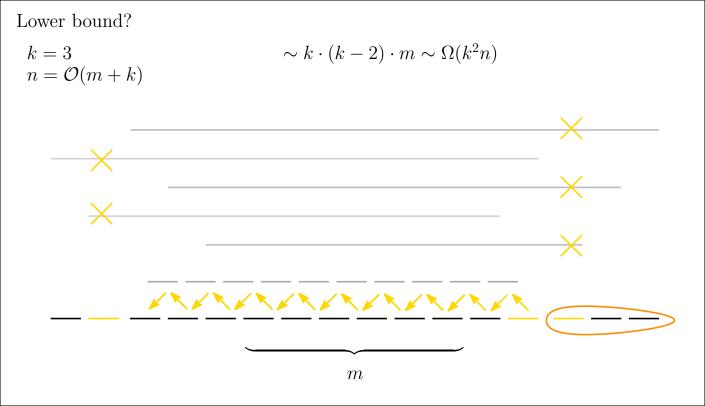


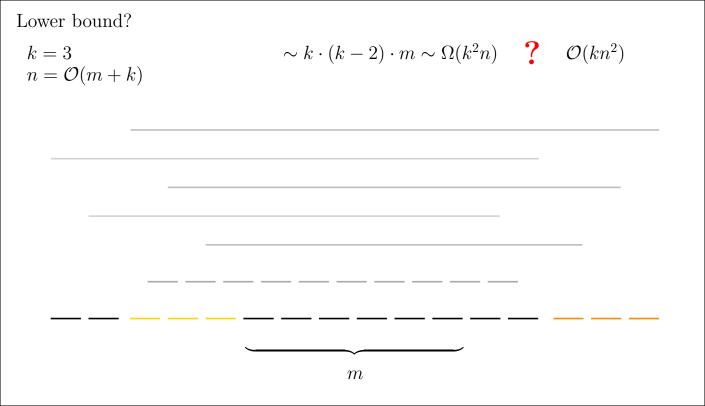


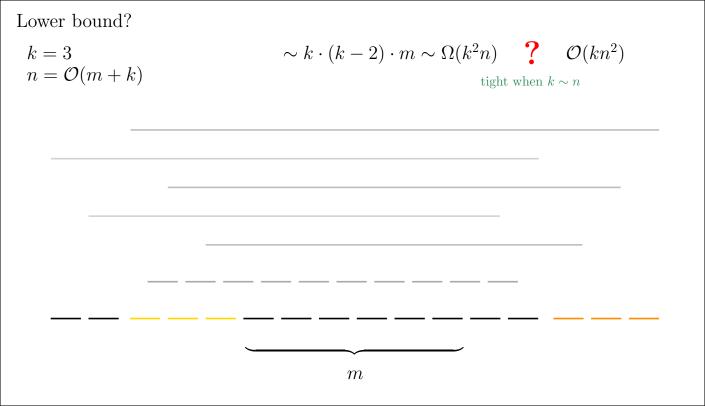


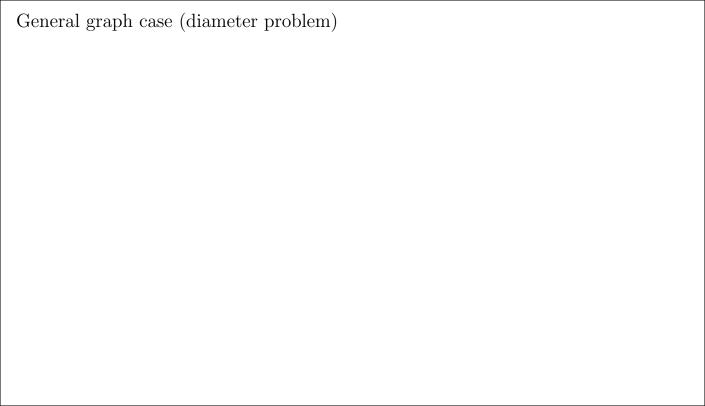












General graph case (diameter problem) $\Omega(n)$ $diam(R_k(G))$ $\mathcal{O}(n^k)$

General graph case (diameter problem)			
$\Omega(n)$	[Hlembotskyi '21]		
$\operatorname{diam}(R_k(G))$	• $k = 2$: linear algorithm (no bound on diameter)		
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 $\Omega(n)$ [Hlembotskyi '21] $\bullet \ k=2: \text{ linear algorithm (no bound on diameter)}$ $\bullet \ k\geqslant 4, \text{ even: } \Omega\left(\left(\frac{n}{k}\right)^{\frac{k}{2}}\right)$

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General graph case (diameter problem)

Open problems:

 $\Omega(n)$ [Hlembotskyi '21] $\bullet \ k=2: \text{ linear algorithm (no bound on diameter)}$

Open problems:

• linear bound for k=2

 $\mathcal{O}(n^k)$

General graph case (diameter problem)

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 $\Omega(n)$ diam $(R_k(G))$

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General graph case (diameter problem)

[Hlembotskyi '21]

• $k \geqslant 4$, even: $\Omega\left(\left(\frac{n}{k}\right)^{\frac{k}{2}}\right)$

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General graph case (diameter problem)

$$\operatorname{diam}(R_k(G))$$

 $\Omega(n)$

 $\mathcal{O}(n^k)$

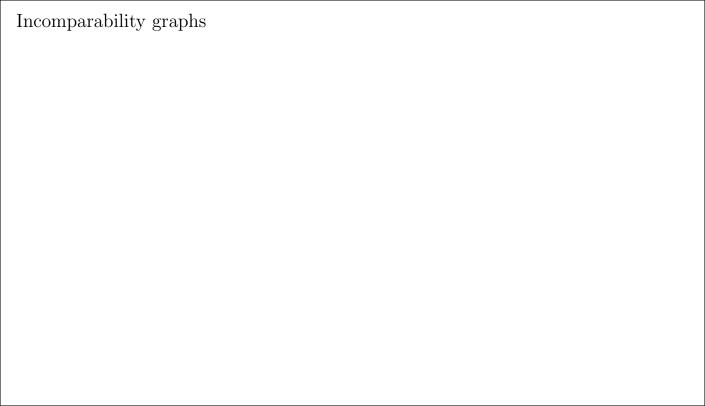
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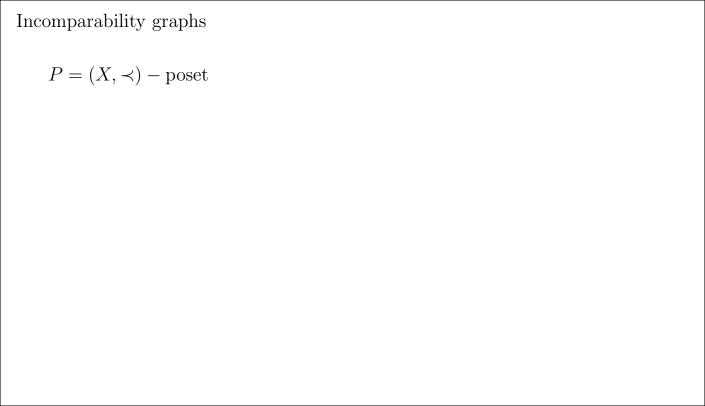
[Hlembotskyi '21]

• k = 2: linear algorithm (no bound on diameter)

- - linear bound for k=2• k = 3?

• is this result tight?





Incomparability graphs $P = (X, \prec) - \text{poset}$ $V(\operatorname{Inc}(P)) = X$

Incomparability graphs

$$P = (X, \prec) - \text{poset}$$

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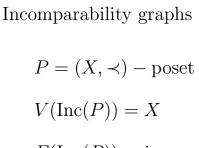
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Example: Interval orders



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$$G$$
 — — —

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$$G \longrightarrow u, v$$

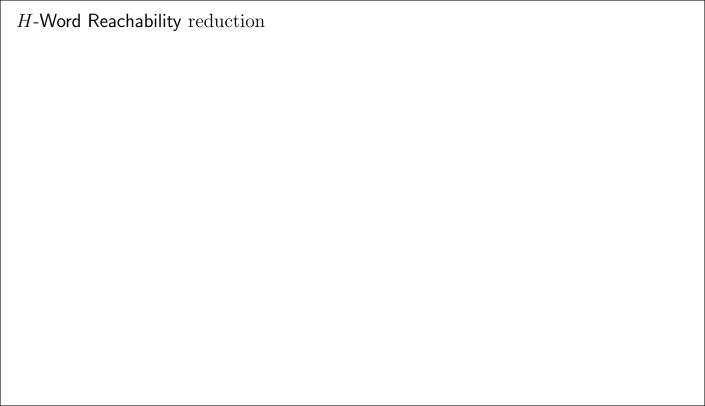
 $u \prec v \Leftrightarrow r(u) < \ell(v)$ $u, v \text{ incomparable } \Leftrightarrow u, v \text{ intersect}$

$$u, v$$
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 $P = (X, \prec) - \text{poset}$ $V(\operatorname{Inc}(P)) = X$ $E(\operatorname{Inc}(P)) = \operatorname{incomparable pairs of vertices}$ Example: Interval orders $u \prec v \Leftrightarrow r(u) < \ell(v)$ u, v incomparable $\Leftrightarrow u, v$ intersect G $\operatorname{Inc}((V(G), \prec)) = G$

Incomparability graphs

Incomparability graphs $P = (X, \prec) - \text{poset}$ Note: chains are independent sets in Inc(P) $V(\operatorname{Inc}(P)) = X$ $E(\operatorname{Inc}(P)) = \operatorname{incomparable pairs of vertices}$ Example: Interval orders $u \prec v \Leftrightarrow r(u) < \ell(v)$ u, v incomparable $\Leftrightarrow u, v$ intersect G $\operatorname{Inc}((V(G), \prec)) = G$



$H ext{-}Word$ Reachability $\operatorname{reduction}$	H - digraph	$a = a_1 a_2 \dots a_j \in V(H)^*$
	a is an \underline{H} -word if $a_i a_{i+1} \in E(H)$ for all i	

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H - digraph

 $a = a_1 a_2 \dots a_j \in V(H)^*$

In: H-words a, b

Out: Can a be transformed to b changing one letter at a time with all intermediate H-words?

 $a ext{ is an } \underline{H}$ -w

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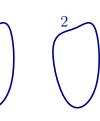
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$$n = |a| = |b|$$
 copies of $V(H)$

a is an H-w

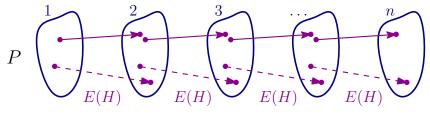
H - digraph

 $a ext{ is an } \underline{H}\text{-word } ext{if } a_i a_{i+1} \in E(H) ext{ for all } i$

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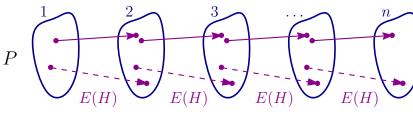
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[Wrochna '18] There exists H such that H-Word Reachability is PSPACE-complete



 $A := \{(a_1, 1), (a_2, 2), \dots, (a_n, n)\}$ $B := \{(b_1, 1), (b_2, 2), \dots, (b_n, n)\}$

n=|a|=|b| copies of V(H)



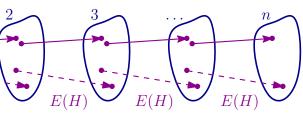
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H - digraph

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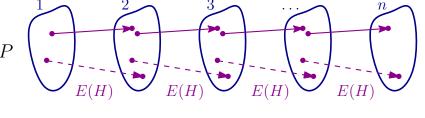
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independent sets in Inc(P)

intermediate words are H-words \iff intermediate sets are independent sets