Defective and clustered choosability of sparse graphs

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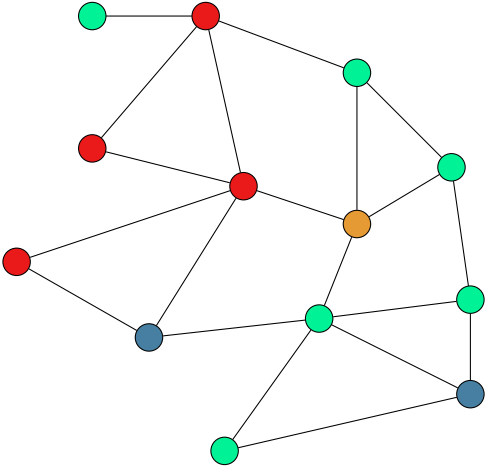
Jagiellonian University

Thursday 27 May, 2021

Based on K. Hendrey and D. R. Wood
"Defective and clustered choosability of sparse graphs"

Improper coloring

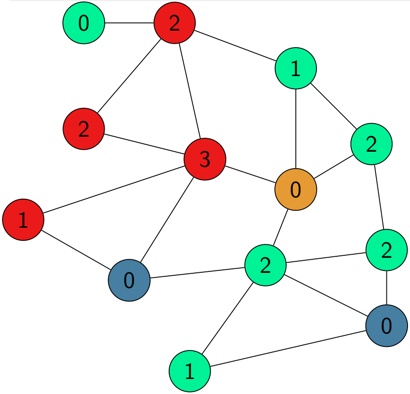
Any assignation of colors to vertices of a given graph.



Defect

Definition

Graph coloring has defect $d \iff$ each vertex is incident to at most d vertices of the same color.

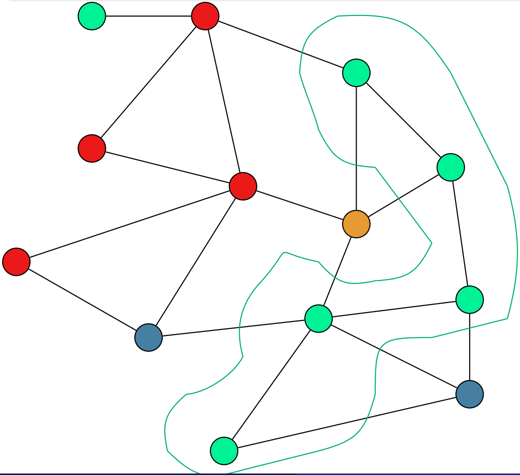


defect=3

Clustering

Definition

Graph coloring has clustering $c \iff$ each monochromatic connected component has size at most c.

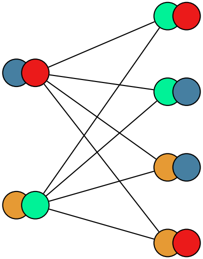


clustering=5

List (proper) coloring

Each vertex v has a list $L(v)$ of allowed colors.

- Graph is L-colorable iff it can be colored properly using only allowed colors, specified by the list $L(v)$ assigned to each vertex v.
- Graph is k-choosable iff it is L-colorable for each L such that $|L(v)| = k$.



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This naturally extends to k-choosability with defect d and k-choosability with clustering c.

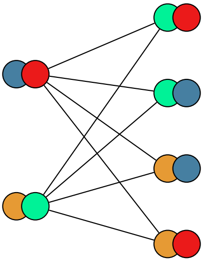


Figure: Above graph is choosable for the given list coloring with defect 1

Sparse graphs

How to measure density of a graph?

- maximum degree over all vertices (Δ)

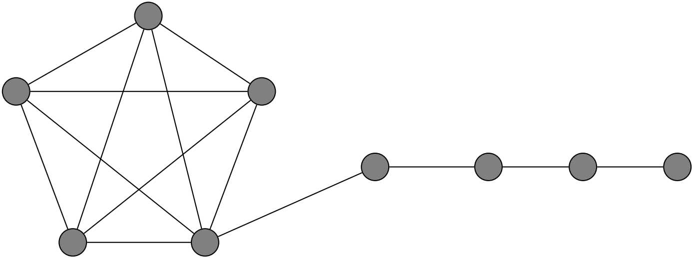


Figure: Graph with $\Delta = 4$

Sparse graphs

How to measure density of a graph?

- maximum degree over all vertices (Δ)
- maximum over all subgraphs from average vertex degree (mad)

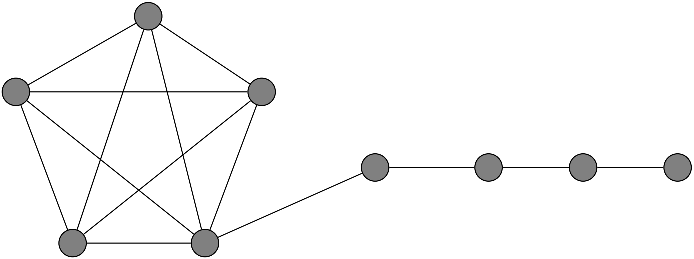


Figure: Graph with $\Delta = 4$ and mad $= 4$

Defective choosability - Pigeonhole Principle Defect Lemma

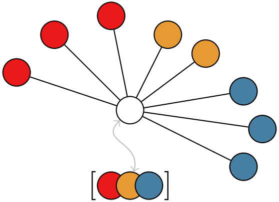
Lemma

We're given a graph G and a list assignment of colors L. If $\forall v : \frac{deg(v)+1}{|L(v)|} \leq d + 1$, *then G is L-colorable with defect d.*

Defective choosability - Pigeonhole Principle Defect Lemma

Lemma

We're given a graph G and a list assignment of colors L. If $\forall v : deg(v) + 1 \leq |L(v)|(d + 1)$, then G is L-colorable with defect d.



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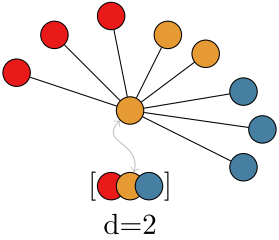
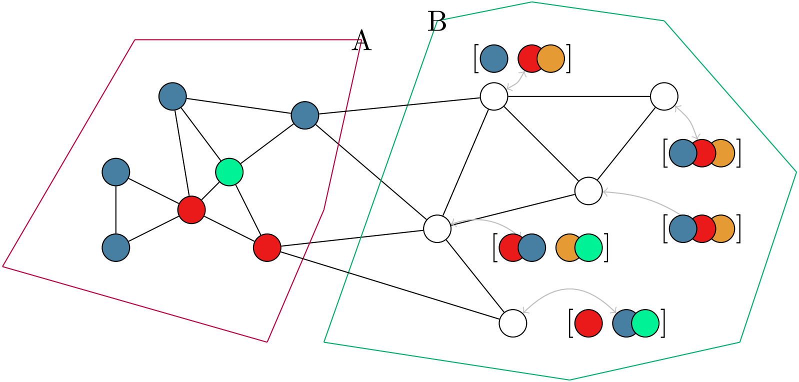


Bounds from Pigeonhole Principle Defect Lemma

For how dense graphs can we prove k-choosability with defect d using above lemma?

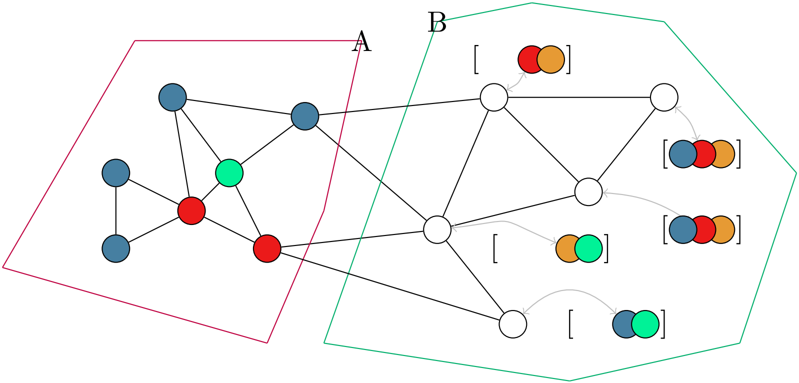
Bounds from Pigeonhole Principle Defect Lemma - idea

Say we colored $A \subset V(G)$ so far.



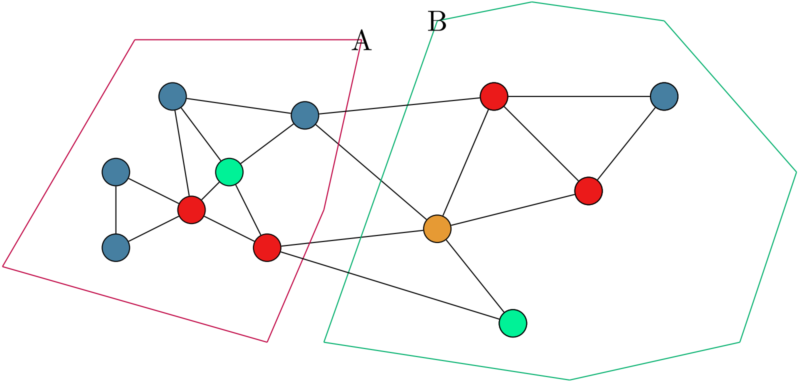
Bounds from Pigeonhole Principle Defect Lemma - idea

Remove from each list assigned to $v \in V(G) \setminus A$ colors used in their neighbours.



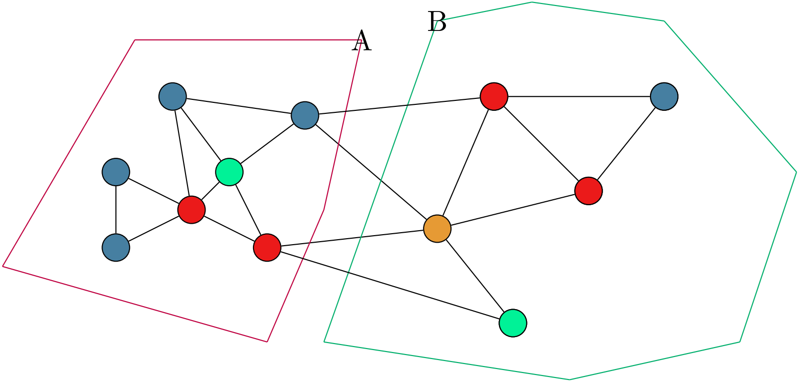
Bounds from Pigeonhole Principle Defect Lemma - idea

If Pigeonhole Principle Defect Lemma works for B after such removal, then we're done.



Bounds from Pigeonhole Principle Defect Lemma - idea

If Pigeonhole Principle Defect Lemma works for *B* after such removal, then we're done.
What if it doesn't work?



Sketch of proof

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- If we can partition $V(G)$ into A, B, such that Pigeonhole Principle Defect Lemma would work for B (after decreasing list size for each $v_b \in B$ by number of its neighbours in A), then apply induction for A and color B.

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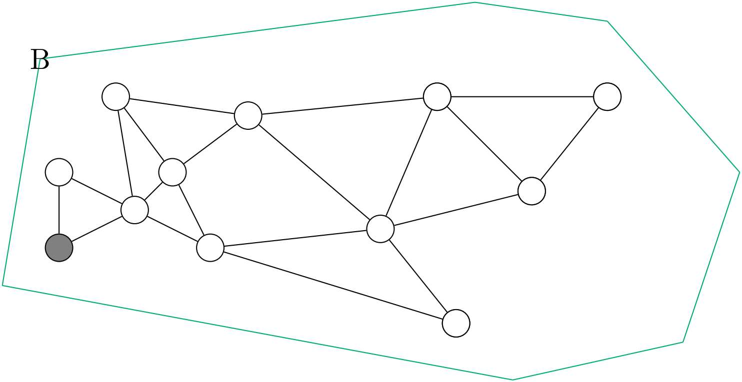
- If we couldn't, then for each partition, there must exist at least one "bad" vertex in B
- "bad" means $(d+1)(k - deg_A(v)) \leq deg_B(v)$

Why the latter case couldn't occur for sparse graphs

- Start with $A_0 = \emptyset$ and pick any bad $v_1 \in B_0$



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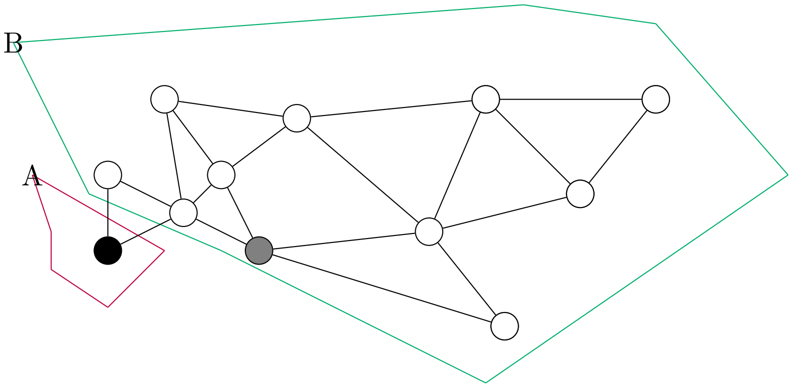

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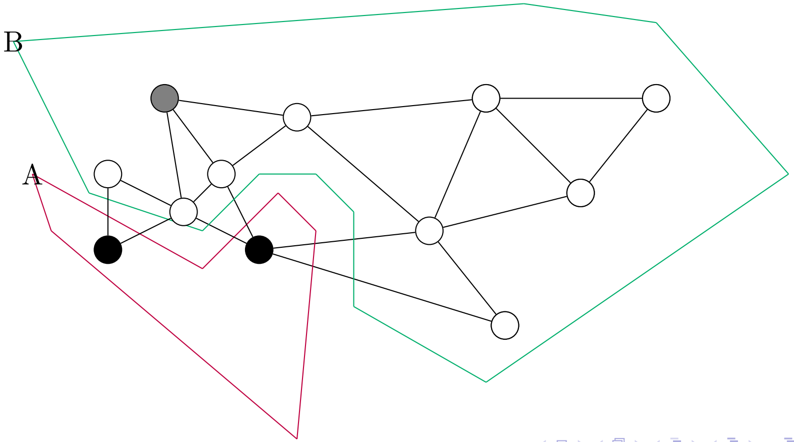

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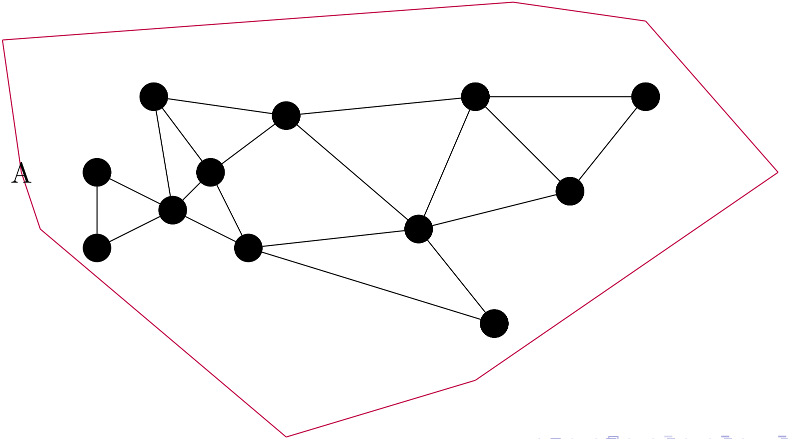
- At the end' we'll have $A = \{v_1, v_2, \ldots, v_n\}$



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- Summing over all vertices, we obtain $k(d+1)|V(G)| - d|E(G)| \leq 2|E(G)|$
- And finally, $2|E(G)|/|V(G)| \geq k(2d+2)/(d+2)$

Defective choosability - result

Theorem

Let G be a graph, such that $mad(G) < \frac{(2d+2)k}{d+2}$. *Then, G is k-choosable with defect d.*

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Let G be a graph, such that $mad(G) < \frac{(2d+2)k}{d+2}$. Then, G is k-choosable with defect d.

This improves previous results [Havet, Sereni, 2006], in which they assumed that $\text{mad}(G) < k + \frac{kd}{k+d}$

Clustering

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How to prove k-choosability with clustering c?

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How to prove k-choosability with clustering c?
We will select k, c and graph sparsity condition such that we will get some bound even if we will allow only path or cycle clusters.

Clustering bounded by a constant

Lemma

If $\forall v : 5|L(v)| \geq 2deg(v) + 2$, *then G has an L-colouring with clustering 6*

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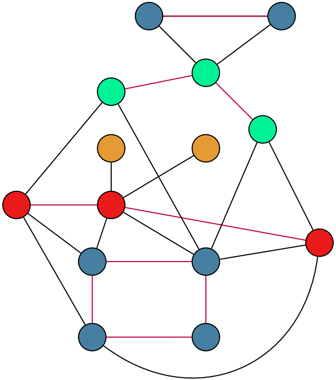

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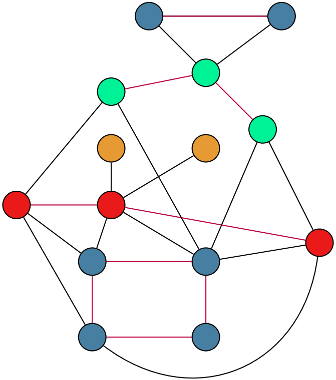

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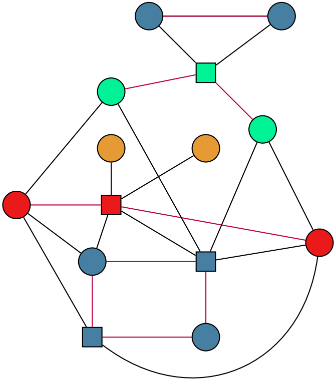

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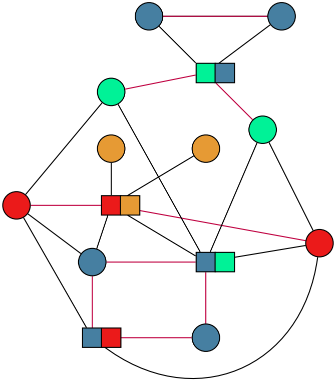

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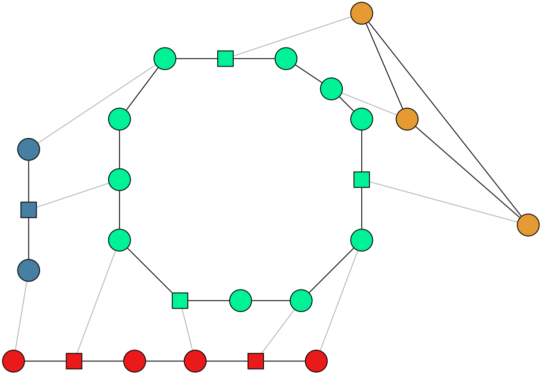
- Then, each monochromatic subgraph has max degree at most 2
- Select largest independent set S of vertices of degree 2 from each cluster
- There exists coloring ϕ' such that each monochromatic component has at most 2 vertices from S
- This implies that each component has at most 6 vertices

Independent transversals

- For proving point fourth point, authors used "independent transversals" method
- There exists independent set S, such that every monochromatic path (cycle) of size at least $a\Delta - b$ contains at least x vertices in S



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Theorem (Direct conclusion)

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Every graph G is $\lfloor \frac{7}{10} mad(G) + 1 \rfloor$ - choosable with clustering 9

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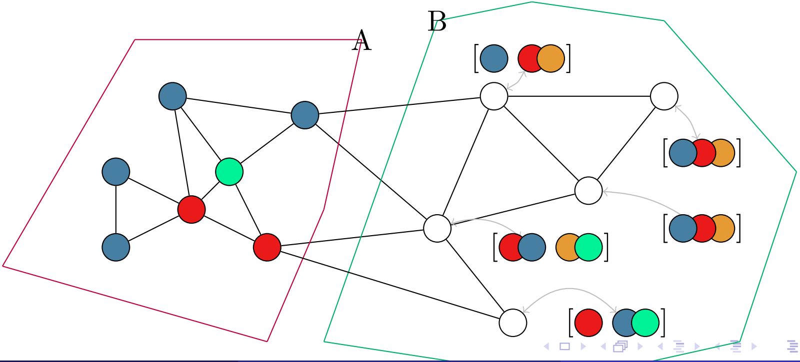

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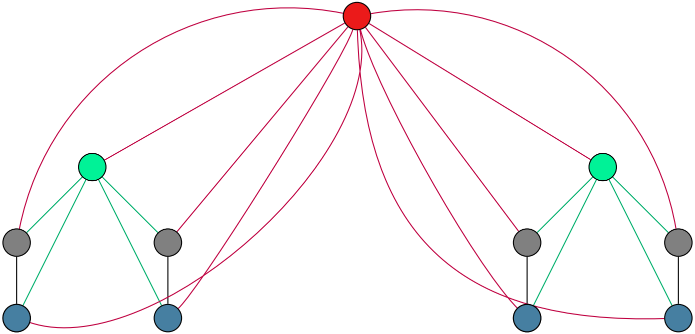
- This gives upper bound to clustered chromatic number of class of graphs with given *mad*
- Best known lower bound: $\lfloor \frac{1}{2} mad(G) \rfloor + 1$



2/3 bound - sketch of proof

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If $\forall v : 3|L(v)| \geq deg(v) + 2$*, then G has an L-colouring with clustering 19Δ - 32*

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Every graph G is $\lfloor \frac{2}{3} mad(G) + 1 \rfloor$ *- choosable with clustering* $57 \lfloor \frac{2}{3} mad(G) \rfloor + 6$

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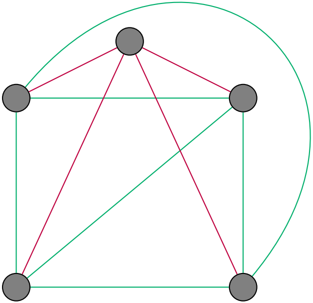
- t-planar graphs (graphs with thickness t)
- k-stack graphs & k-queue graphs
- any class of graphs with bounded mad, but with unbounded Δ

Applications - biplanar graphs

Definition

t-planar graph is a graph $G = (V, E)$ whose edges can be partitioned to t sets $E_1, \ldots E_t$, such that $G_i = (V, E_i)$ is planar



Applications - biplanar graphs

Every biplanar graph has $mad \leq 12$ (follows from Euler's formula)

- Results from previous papers show, that biplanar graphs are k-choosable with defect d for:

list lengths (k)	5	6	7	8	9	10	11
defect (d)	36	19	12	9	6	4	2

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- 2/3 bound theorem tells that they are 8-choosable with clustering 405
- It is open, whether biplanar graphs are 6 or 7-choosable with bounded clustering

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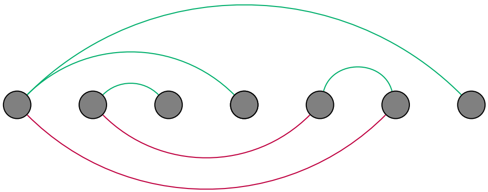
- $4t$-choosable with clustering $228t$ - 51

Applications - k-stack graphs

Definition

k-stack graph is a graph whose edges can be partitioned into k sets $E_1, \dots E_k$ and whose vertices can be ordered $v_1, \dots v_n$, such that for no two pairs $(v_a, v_b), (v_c, v_d)$ of edges in E_i they cross, which means $a < c < b < d$



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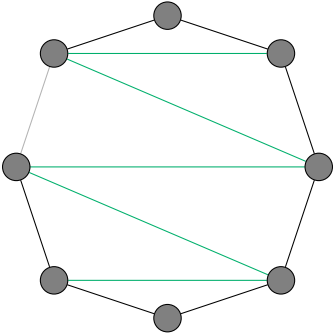
Every k-stack graph has $mad \leq 2k+2$, thus is:

- $\lfloor (3k+4)/2 \rfloor$-choosable with clustering 2,



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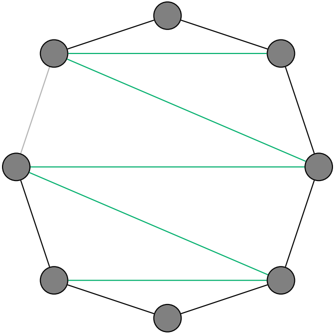

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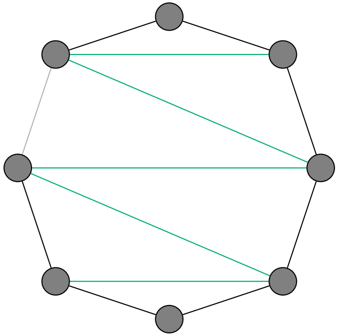
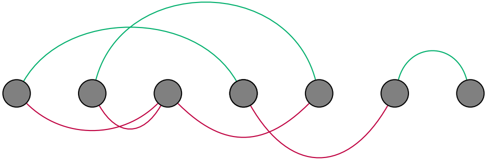
- $\lfloor (4k+6)/3 \rfloor$-choosable with clustering $76k+53$



Applications - k-queue graphs

Definition

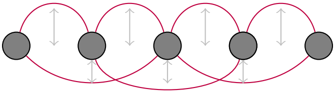
k-queue graph is a graph whose edges can be partitioned into k sets $E_1, \dots E_k$ and whose vertices can be ordered $v_1, \dots v_n$, such that for no two pairs $(v_a, v_b), (v_c, v_d)$ of edges in E_i they are nested, which means $a < c < d < b$



Applications - k-queue graphs

Every k-queue graph has $mad \leq 4k$, thus is:

- $3k$-choosable with clustering 2,



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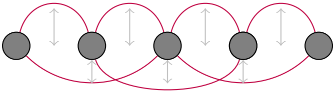
- $3k$-choosable with clustering 2,
- $\lfloor (14k + 4)/5 \rfloor$-choosable with clustering 9,



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Every k-queue graph has $mad \leq 4k$, thus is:

- $3k$-choosable with clustering 2,
- $\lfloor (14k + 4)/5 \rfloor$-choosable with clustering 9,
- $\lfloor (8k + 2)/3 \rfloor$-choosable with clustering $152k$-13